Polar alignment by the single-star drift method

Introduction

Internet searches for polar alignment of a telescope mount using the drift method yields many hits. The tutorials I have seen talk about measuring drift of star trails for stars in two directions, one near the equator at the meridian, the other east or west near the horizon. Two directions are needed because each is only sensitive to the mount's polar misalignment in one direction. The need for two stars and an iterative procedure that goes back and forth between them it is said makes the method cumbersome and time consuming. For these reasons this method is not very attractive (I haven't actually used the method).

As you know, a camera sitting on a tracking, perfectly aligned mount is stationary with respect to the stars and will see no tracks, neglecting refraction. With the polar axis pointed away from the pole by some angular distance, the camera sees tracks in proportion to that misalignment. Conceptually it important to understand that the tracks arise from fact the camera is rotating with respect to stars, and that this rotation has an axis determined by the mount's polar-axis misalignment. Now the crucial point is to recognize that that axis of rotation is perpendicular to the Earth's north pole, i.e., an axis through the equator somewhere – where depending on in what direction the mount's axis is miss-pointed. When the camera is pointing towards or near that axis of rotation, it cannot see trails. It is like a camera fixed to the Earth pointing at the pole. Thus the need for two camera orientations when using the drift method of alignment: each cannot see trails for misalignments causing rotations whose axis is on the camera's line of sight.

What is missed by the authors of these drift-alignment tutorials is that there is one camera orientation that can see trails regardless of the polar-axis misalignment — this being the pole, simply because the axis of rotation is always on the equator where the camera's line of sight is not. Furthermore, the direction of the star trails at and around the pole reflects the orientation on the equator of the axis of rotation caused by the misalignment. Thus camera exposures of the pole alone are sufficient to the perform polar alignment by the this drift method.

I've said a number of things about the nature of the camera's rotation with respect to the stars on a tracking mount when there is polar axis misalignment, that require justification of a mathematical nature. I provide that in a later section. There are sound and compelling arguments in that section, but I realize that most readers of this article want to see evidence of an empirical nature before trusting such arguments. For this reason I'll first show my test of the basic idea of drift alignment using polar photos and later give the mathematical argument to those who are interested.

There is also a short section that talks about rotation of the field of view about a guide star when tracking and guiding.

The test

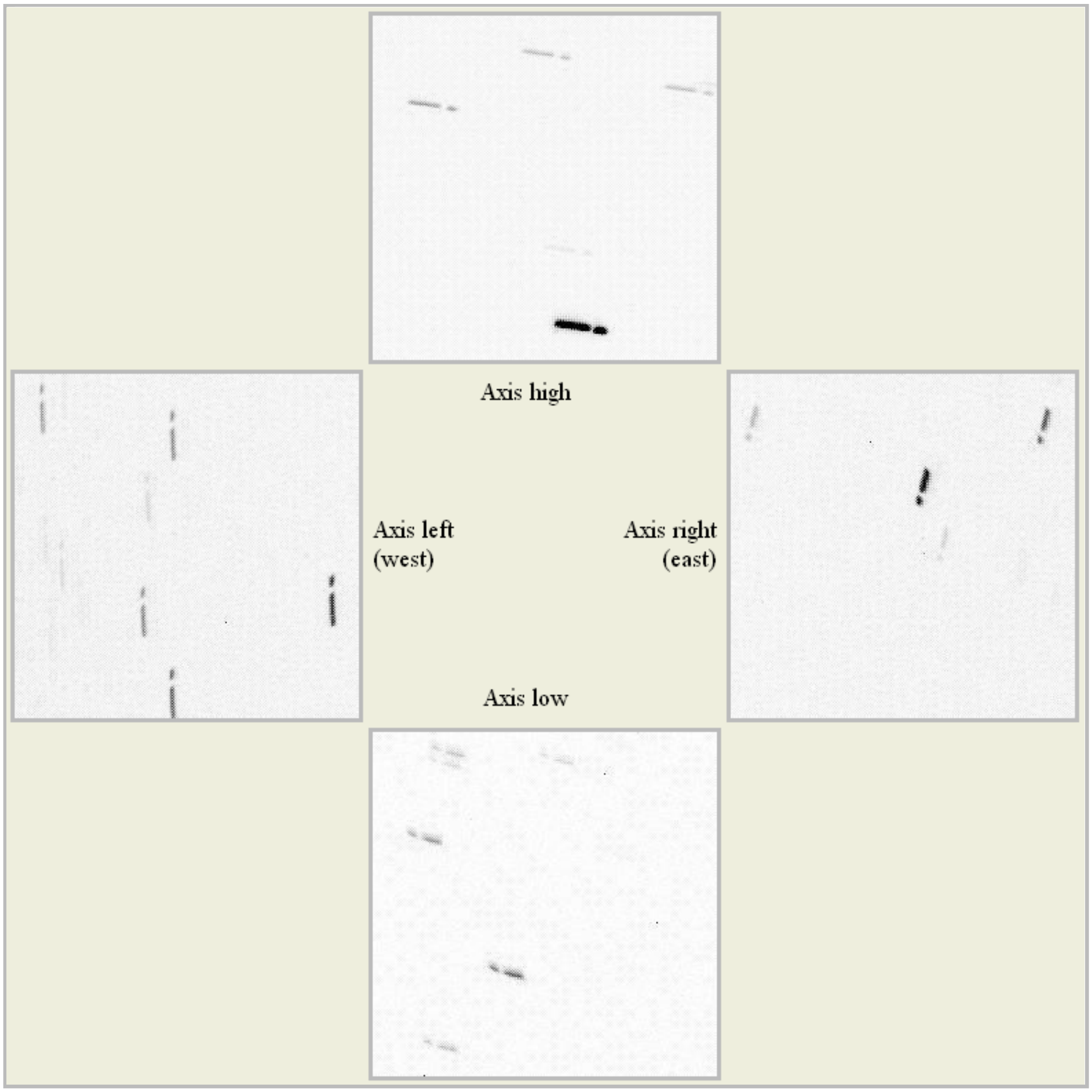
The goal of this test is to measure the dependence of the direction of star trails in the viscinity of the pole while tracking, on the pointing of the polar axis. The possibilities are that there is none; that it depends on where in relation to the pole the star is located as well as the pointing of the polar axis; or that the trails are the same everywhere in the field of view, regardless of the location of the pole, but depending of the pointing of the polar axis. It is only the last case that is useful for polar alignment, the only one that gives a definite direction of star trails as a function of pointing alone, and one where star-trail pointing can be used to infer polar-axis pointing.

The equipment is a Canon 350D, a Canon 75-300 mm zoom lens with the lens set at 300 mm, and an Orion Atlas mount. The left-right field of view of the camera and lens is 4.2 degrees. When pointing at the pole, the camera was oriented on top of the mount so that it was approximately upright, i.e., left in each image is west, right is east, and up and down are toward and away from the zenith, respectively. I first took four pictures, each about three minutes of exposure time with the polar axis pointed a few degrees above, below, left, and right of the pole. To determine the direction of drift while tracking, the aperture was covered for 30 seconds starting 30 seconds into each exposure. The idea of blocking the aperture to show the direction of trailing of the stars came from the internet somewhere, and worked well in these exposures.

Below is shown the full field of view, at one-third scale when expanded, of the image taken with the polar axis pointed a few degrees left of the pole. This is the only photo of the four showing Polaris. The north pole is a bit off the right edge. All of these photos are shown with inverted colors (printing dark photos empties ink cartridges). They also are of raw pixels showing evidence of the Bayer matrix.

The photo above clearly shows a downward drift of the stars. The crucial thing to note about that drift is that it is uniform over the field of view, and that it does not depend discernibly on angular distance to the north pole. All four of the photos show this uniform drift over the field of view.

Crops of the four photos at the full pixel scale to show more clearly the trails are shown below. Each is oriented with up being up, like with the photo above. The arrangement of the four photos parallels the side of the pole the mount's axis was pointed: approximately above, below, left (west), and right (east) of the pole.



Note the apparent counter-clockwise rotation in the arrangement of photo of the drifts with respect to direction of axis misalignment. From these photos it is evident that the direction of the north pole is an angular distance in a direction about 90 degrees counter clockwise from the direction of the star trails.

We can go further with the data available here. Given that we know the length of the trails in pixels (32 pixels in the photo to the left of the pole), the camera's pixel size, the lens focal length, and the exposure time, we can compute the rate of drift of the stars. This comes to

rate = 32 pixels × 6.4 µm/pixel ÷ 300 mm fl ÷ 3 minutes
= 19 degrees per day
= 1/19 sidereal rate

I'll show in the math section that this drift rate is closely approximated by the product of the sidereal rate and the polar-axis misalignment in radians.

star trails' rate = misalignment (in radians) × sidereal rate

Thus the measured rate tells us the angular distance from the mount's axis to the pole. From data from the first photo, the pole is about three degrees right of the mount's polar axis.

I'll further show below that the angle between the star trails and the direction from the mount's polar axis to the north pole is closely approximated by 90 degrees, and not just a guess supported by the photos. As you can see, data provided by a single photo can be very specific about how to reorient the mount's polar axis to align with the Earth's axis. This is the method I suggest using as an alternative to other drift-alignment methods.

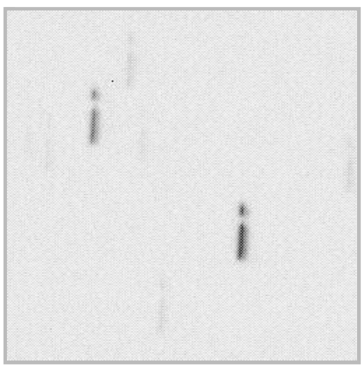
In a nutshell:

1. Set up your telescope or camera on a tracking mount approximately pointing towards the pole.
2. Determine the orientation of the camera's images with respect to changes of the mount's polar axis.
3. Photograph the pole area with an exposure of convenient length while covering the aperture during part of the exposure in such a way that the direction of trails can be seen (such as mentioned above).
4. From the star trails on the image, compute the angular rate at which they are moving, and divide by the sidereal rate. This is the angle in radians between the Earth's axis and the mount's polar axis.
5. Move the mount's polar axis in the direction that is counter clockwise from, and at right angle to, the direction of the star trails, and by an angle in that direction determined in the previous step.
6. Repeat steps 3 to 5 as necessary.

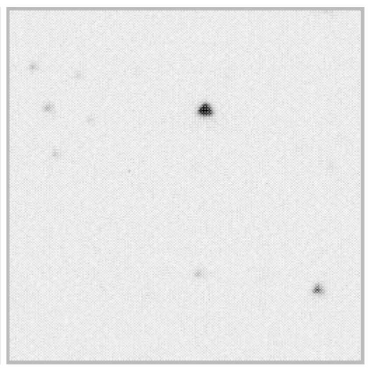
A lot of details obviously are not spelled out. (Remember Steve Martin: How do you make a million dollars and then not pay any taxes? ...) Exactly how fast this method can be made to work depends on one's ingenuity.

Using the method to align the polar axis bore out these results. A couple of nights later I applied one iteration of this method to the mount and got the following before and after photos. These are three-minute exposures with the same 300-mm lens.

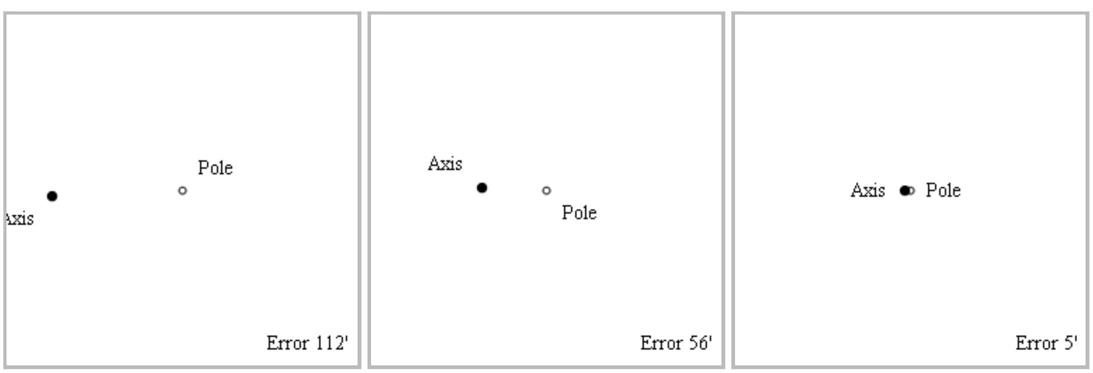
Before



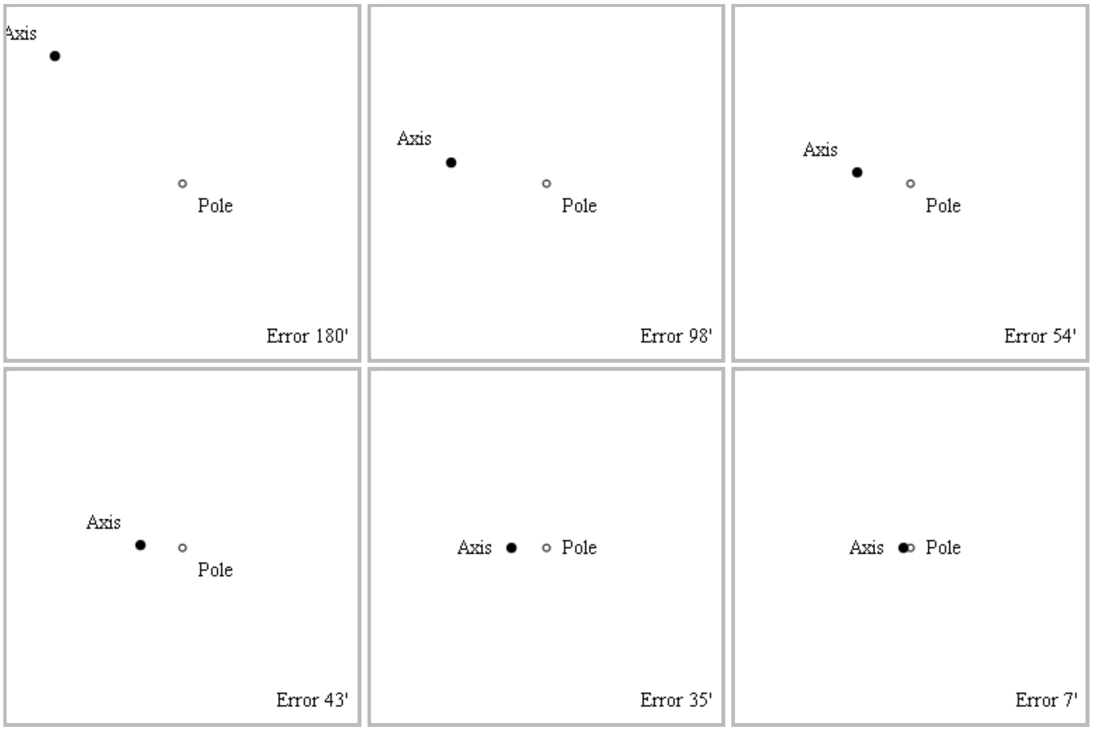
After



This sequence of corrections was taken a couple of days later using an 203-mm f/4.7 telescope, where the positions inferred from the star trails are calculated as described above. The zenith is up in these figures.



This is a later sequence.



I am slowly getting better at aligning the polar axis. This last sequence required about 15 minutes to complete. Unfortunately, I haven't been able to do better than this due to poor equipment. What is needed is a better camera focus, longer exposures, longer focal length, and finer mount positioning controls (or a more delicate touch). Or perhaps just more practice.

The math

This section is mathematical, and as such it is not for everyone. The methods employed are well known, and I am not claiming anything is original.

The problem is to understand how miss-pointed polar axes generate star trails. This is about rotations, or members of the orthogonal group [2] in three dimensions, SO(3) [3] represented by 3x3 matrices whose transposes are their inverses. The matrix of a rotation about the z axis through an angle θ looks like this:

$$R_z(\theta)=\begin{bmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}$$

Rotations about the x and y axes are derived from this expression by cyclically permuting the indices. Rotations matrices can be expresses as exponential of their generators. For example, R_z can be expressed as

$$R_z(\theta)=e^{\theta\sigma_z}$$

where σ_z , called the generator of R_z , is the matrix

$$\sigma_z=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

So if the z axis is taken to be the Earth's axis, the Earth's rotation is represented by the matrix $e^{\omega t\sigma_z}$, where the Earth's rotation rate is $\omega=2\pi/$ sidereal day.

The other two generators, for x and y , are

$$\sigma_x=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}$$

and

$$\sigma_y=\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}$$

These three matrices form a vector of matrices

$$\vec{\sigma}=[\sigma_x,\sigma_y,\sigma_z]$$

from which an arbitrary rotation R can be constructed from three parameters arranged in a vector $\vec{v}=\{a,b,c\}$,

$$R=e^{\vec{v}\cdot\vec{\sigma}}$$

using the inner product $\cdot$. The direction of the vector $\vec{v}$ is the axis of rotation of R , and its magnitude is the angle through which R rotates. The exponential of the matrix argument is defined by the power series for the exponential function.

The problem at hand is to compute how the positions of stars as detected by a camera on a tracking mount that is misaligned change with time. We had talked about the coordinate transformation from celestial coordinates (relative to the stars) to Earth coordinates (relative to the fixed Earth). Written explicitly, that coordinate transformation can be expressed like this.

$$\begin{bmatrix}x\\y\\z\end{bmatrix}_{Earth}=e^{\omega t\,\sigma_z}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}_{celestial}$$

But now the mount axis is offset from the Earth's axis by some angle δ . We can think of that angle as the mount's axis having been rotated relative to the earth by the angle. That rotation has an axis that is uniquely defined by the perpendicular to the plane defined by the Earth's axis and the mount's polar axis. Note that this axis of rotation is on the equator because is has to be perpendicular to the polar axis. Call the unit vector corresponding to this axis of rotation $\vec{u}$ and define the matrix $u=\vec{u}\cdot\vec{\sigma}$. Now we have a rotation that maps Earth coordinates (fixed to the Earth) to coordinates relative to the mount (not tracking yet).

$$\begin{bmatrix}x\\y\\z\end{bmatrix}_{mount}=e^{\delta\,u}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}_{Earth}$$

Now we have to talk about the mount's tracking and its matrix. Tracking is there to undo the Earth's rotation when the mount is perfectly aligned, so it has an equal and opposite rotation rate. As such, its matrix transforms from the mount coordinates to the camera coordinates like this.

$$\begin{bmatrix}x\\y\\z\end{bmatrix}_{camera'}=e^{-\omega t\,\sigma_z}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}_{mount}$$

Notice the minus sign in front of ω , and also the prime above the camera. The latter is added because I want to add another fixed (not time dependent) coordinate transformation to the unprimed camera coordinates. It shifts camera coordinates slightly but otherwise does nothing; it is added because it makes the final result prettier.

$$\begin{bmatrix}x\\y\\z\end{bmatrix}_{camera}=e^{-\delta\,u}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}_{camera'}$$

The map that tells us how the camera sees the stars is the concatenation of these four maps.

$$\begin{bmatrix}x\\y\\z\end{bmatrix}_{camera}=e^{-\delta\,u}\cdot e^{-\omega t\,\sigma_z}\cdot e^{\delta\,u}\cdot e^{\omega t\,\sigma_z}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}_{celestial}$$

It is the concatenated exponentials, call it the matrix M , that performs this coordinate transformation that we will look at more closely.

$$M=e^{-\delta\,u}\cdot e^{-\omega t\,\sigma_z}\cdot e^{\delta\,u}\cdot e^{\omega t\,\sigma_z}$$

Notice that if the misalignment angle δ is zero, then the two δ matrices go away (unit matrix) and the matrices containing the Earth's rotation rate ω also cancel each other because they are inverses of each other. In that case $M=$ the unit matrix, i.e., perfect tracking.

From here, we note that the angles δ and ωt are both small for misalignments and exposure times of interest. The angle δ ranges anywhere from a few degrees (1/20 radian) to millidegrees ($\sim 10^{-5}$ radian), while an exposure of, say, 10 minutes is an angle ωt of 2.5 degrees (0.04 radian). So it is feasible to expand M as a power series in δ and ωt and truncate all but terms first order is δ and ωt , plus terms with $\delta\times\omega t$. It is the last that has the rotation we need to know. Alternatively, we can apply the Campbell-Baker-Hausdorff theorem [4] to each matrix product in turn, again keeping low-order terms. Using the latter approach we get the expression for M (to low order it is a simpler calculation than it might seem):

$$M=e^{\delta\,\omega t\,[u,\sigma_z]}+\ldots$$

where the brackets in the exponent are the Lie bracket of the two matrices, i.e., $[u,\sigma z]=u\cdot\sigma z-\sigma z\cdot u$. We are quite close to the answer now. Remember that the matrix $u=\vec{u}\cdot\sigma$, and the σz can be expressed $\sigma z=\vec{z}\cdot\sigma$, where $\vec{z}$ is the unit vector $\vec{z}=\{0,0,1\}$. It is a useful fact about the Lie algebra of SO(3) that

$$[u,\sigma z]=(\vec{u}\times\vec{z})\cdot\sigma$$

quite generally, where $\vec{u}\times\vec{z}$ is the vector cross product of $\vec{u}$ and $\vec{z}$, which means that

$$M=e\delta\,\omega t\,(\vec{u}\times\vec{z})\cdot\sigma+\ldots$$

So the axis of rotation of M is $\vec{u}\times\vec{z}$, and the angle through which M rotates is $\delta\,\omega t$. Call the axis of rotation

$$\vec{v}=\vec{u}\times\vec{z}$$

This vector is perpendicular to $\vec{z}$, and so must be on the celestial equator. That it is on the equator ensures that the camera will see rotation at the pole regardless of its direction on the equator. Its position on the equator is determined by the fact that it is also perpendicular to $\vec{u}$.

This is the answer we are looking for. The axis-of-rotation vector $\vec{v}$ and the angle of rotation $\delta\times\omega t$ characterize the rotation of the stars seen by the camera on the tracking mount.

It should be noted that the terms of M higher order in the misalignment δ and exposure duration ωt have the effect that M is not exactly characterized by a rotation axis, and that such a characterization is accurate to a given degree only for small enough misalignments and short enough exposures. It is perhaps the higher-order terms in ωt that give the tracks in the photos above their curvature, which is clearly evident. But those photos also show that for the misalignments and exposure times used to take those photos (quite large and not very large, respectively), those higher-order effects are still small.

Rotation about the guide star during guided exposures

Guiding on a guide star fixes the position in the telescope's field of view of the star, often to the sub-pixel level. While the guide star's position is fixed to a high degree, other stars in the field of view need not be fixed. In fact, the field of view rotates about the guide star at a rate that depends on the degree of misalignment of the mount's polar axis. It is like photographing the celestial pole without guiding and seeing star trails, except that the rate of rotation is much reduced. I am going to give the result in this section, but mercifully without giving a detailed derivation.

Like before, ω is the Earth's rotation rate, 'dec' is the guide star's declination, and δ is the mount's polar-axis mialignment in radians. Then the rotation of the field of view has the rate.

$$\text{rate}=\delta/\cos(\text{dec})\times\omega$$

The rotation rate is least near the celestial equator, while greatest near the pole. It is smaller than the Earth's rate by a factor given by the misalignment in radians (and the declination). So if the misalignment is one degree, then the rate is the Earth's rotation rate reduced by the factor 1/57/cos(dec).

I use my telescope as an example. It has 950-mm focal length, its field of view has 0.7-degree radius, and the camera has 6.4-µm pixels. When photographing the pole centered in the field of view without guiding or tracking, stars at the edge of the field drift a distance of one pixel in 7.6 seconds. In contrast, with guiding and tracking and one-degree misalignment, equatorial stars at the edge of the field of view centered on a guide star drift a distance of one pixel in 7.2 minutes.

Note that this formula breaks down as the declination approaches the pole and should not be used there. To use this formula reliably, the guide star's declination should differ from 90 degrees by a lot more than how far the mount's axis differs from 90 degrees declination ($\pi/2-\text{dec}\gg\delta$).

A final comment

Judging from the tutorials on polar alignment around today, there is a distinct aversion to the use of stars near the pole for measurement of drift tracks on tracking mounts. This is no doubt a misconception brought about by analogy with the lack of apparent motion of stars near the pole as viewed from a reference fixed to the rotating Earth. But, as was demonstrated here, a camera on a misaligned tracking mount sees quite a different kind of rotation compared to someone fixed to the Earth, where the pole stars are in motion as much as any others elsewhere in the sky. This is the fact that makes this method of polar-axis alignment possible.

I further think that this method could be a high-precision alignment method, particulaly among those already with the talent for the two-star drift alignment method. When algorithms are used that quantitatively privide corrections to the mounts polar axis, such as those employed by some go-to mounts and third-party software, convergence may be quite fast. One can also employ very long exposures when making the very fine adjustments, or even two exposures separated by an hour or more, and use the blink method to sense motion. Then an axis correction can be computed and applied to the mount.

This picture may be overly optimistic, especially in light of the views expressed in reference 1. But I am sure both the speed and the practically achievable accuracy of polar-axis alignment can be pushed using this method.

References

1. For another perspective on drift alignment, see http://www.brayebrookobservatory.org/BrayObsWebSite/HOMEPAGE/BRAYOBS PUBLICATIONS.html.
2. Wikipedia, *Orthogonal group*, http://en.wikipedia.org/wiki/Orthogonal_group.
3. Wikipedia, *Rotation group SO(3)*, http://en.wikipedia.org/wiki/Rotation_group_SO(3).
4. Wikipedia, *Baker-Campbell-Hausdorff formula*, http://en.wikipedia.org/wiki/Campbell-Baker-Hausdorff_formula.

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