Design of Engineering Experiments Chapter 6 – The 2^k Factorial Design

- Text reference, Chapter 6
- **Special case** of the general factorial design; *k* factors, all at two levels
- The two levels are usually called **low** and **high** (they could be either quantitative or qualitative)
- Very widely used in industrial experimentation
- Form a basic “building block” for other very useful experimental designs (DNA)
- Special (short-cut) methods for analysis
- We will make use of Design-Expert

The Simplest Case: The 2^2

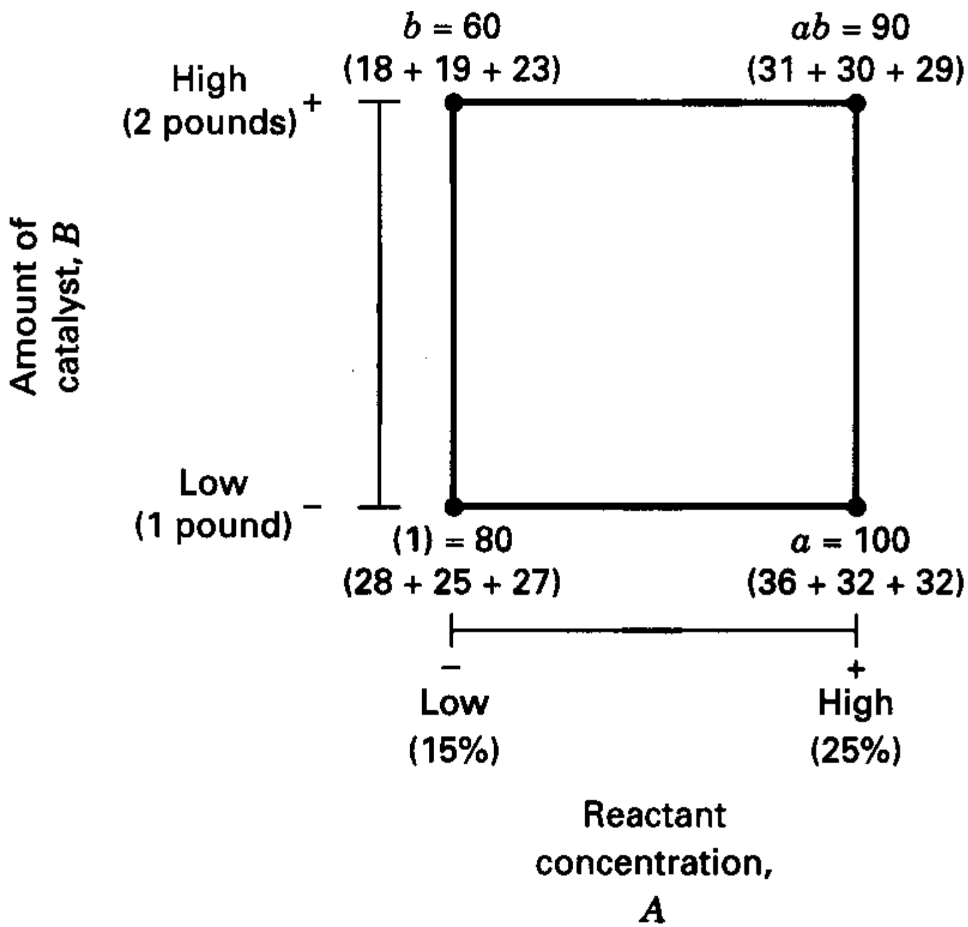


Figure 6-1 Treatment combinations in the 2^2 design.

"-" and "+" denote the low and high levels of a factor, respectively

Low and high are arbitrary terms

Geometrically, the four runs form the corners of a square

Factors can be quantitative or qualitative, although their treatment in the final model will be different

Chemical Process Example

Factor		Treatment	Replicate			
A	*B*	Combination	I	II	III	Total
−	−	*A* low, *B* low	28	25	27	80
+	−	*A* high, *B* low	36	32	32	100
−	+	*A* low, *B* high	18	19	23	60
+	+	*A* high, *B* high	31	30	29	90

A = reactant concentration, *B* = catalyst amount, *y* = recovery

Analysis Procedure for a Factorial Design

- Estimate factor **effects**
- **Formulate** model
  - With replication, use full model
  - With an unreplicated design, use normal probability plots
- Statistical **testing** (ANOVA)
- **Refine** the model
- Analyze **residuals** (graphical)
- **Interpret** results

Estimation of Factor Effects

$$A = \bar{y}_{A^+} - \bar{y}_{A^-}$$
$$= \frac{ab + a}{2n} - \frac{b + (1)}{2n}$$
$$= \tfrac{1}{n}[ab + a - b - (1)]$$

$$B = \bar{y}_{B^+} - \bar{y}_{B^-}$$
$$= \frac{ab + b}{2n} - \frac{a + (1)}{2n}$$
$$= \tfrac{1}{n}[ab + b - a - (1)]$$

$$AB = \frac{ab + (1)}{2n} - \frac{a + b}{2n}$$
$$= \tfrac{1}{n}[ab + (1) - a - b]$$

See textbook, pg. 221 For **manual** calculations

The effect estimates are:
A = 8.33, B = -5.00, AB = 1.67

Practical interpretation?

Design-Expert analysis

Estimation of Factor Effects
Form Tentative Model

	Term	Effect	SumSqr	% Contribution
Model	Intercept			
Model	A	8.33333	208.333	64.4995
Model	B	-5	75	23.2198
Model	AB	1.66667	8.33333	2.57998
Error	Lack Of Fit	0	0	
Error	P Error	31.3333	9.70072	

Lenth's ME	6.15809
Lenth's SME	7.95671

Statistical Testing - ANOVA

Response: Conversion
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**
Model	291.67	3	97.22	24.82	0.0002
A	*208.33*	*1*	*208.33*	*53.19*	*< 0.0001*
B	*75.00*	*1*	*75.00*	*19.15*	*0.0024*
AB	*8.33*	*1*	*8.33*	*2.13*	*0.1828*
Pure Error	31.33	8	3.92		
Cor Total	323.00	11			

Std. Dev.	1.98	R-Squared	0.9030
Mean	27.50	Adj R-Squared	0.8666
C.V.	7.20	Pred R-Squared	0.7817
PRESS	70.50	Adeq Precision	11.669

The *F*-test for the "model" source is testing the significance of the overall model; that is, is either *A*, *B*, or *AB* or some combination of these effects important?

Statistical Testing - ANOVA

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	27.50	1	0.57	26.18	28.82	
A-Concent	4.17	1	0.57	2.85	5.48	1.00
B-Catalyst	-2.50	1	0.57	-3.82	-1.18	1.00
AB	0.83	1	0.57	-0.48	2.15	1.00

General formulas for the standard errors of the model coefficients and the confidence intervals are available. They will be given later.

Refine Model

Response: Conversion

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**
Model	283.33	2	141.67	32.14	< 0.0001
A	*208.33*	*1*	*208.33*	*47.27*	*< 0.0001*
B	*75.00*	*1*	*75.00*	*17.02*	*0.0026*
Residual	39.67	9	4.41		
Lack of Fit	*8.33*	*1*	*8.33*	*2.13*	*0.1828*
Pure Error	*31.33*	*8*	*3.92*		
Cor Total	323.00	11			

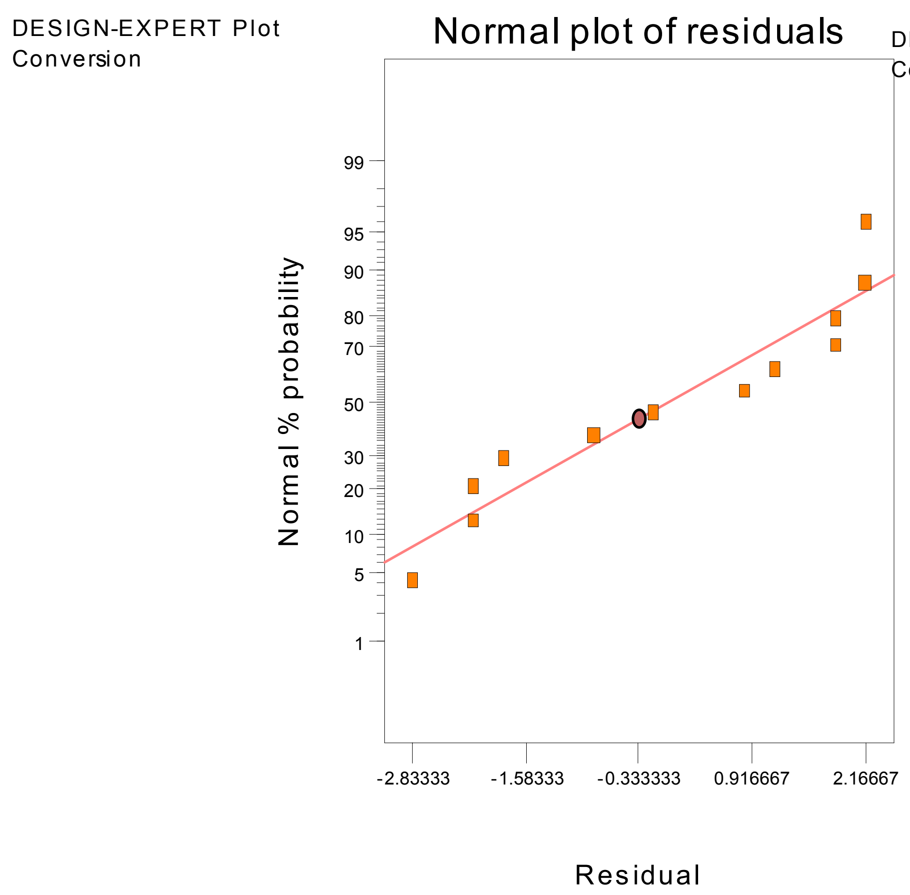
Std. Dev.	2.10	R-Squared	0.8772
Mean	27.50	Adj R-Squared	0.8499
C.V.	7.63	Pred R-Squared	0.7817
PRESS	70.52	Adeq Precision	12.702

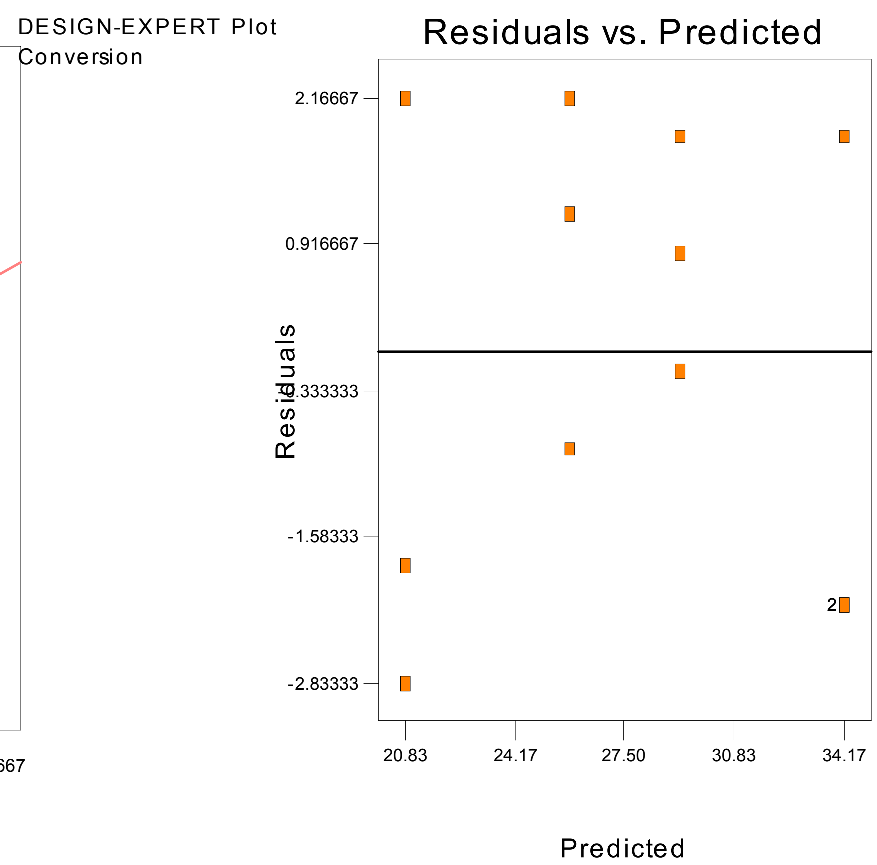
There is now a residual sum of squares, partitioned into a “lack of fit” component (the AB interaction) and a “pure error” component

Regression Model for the Process

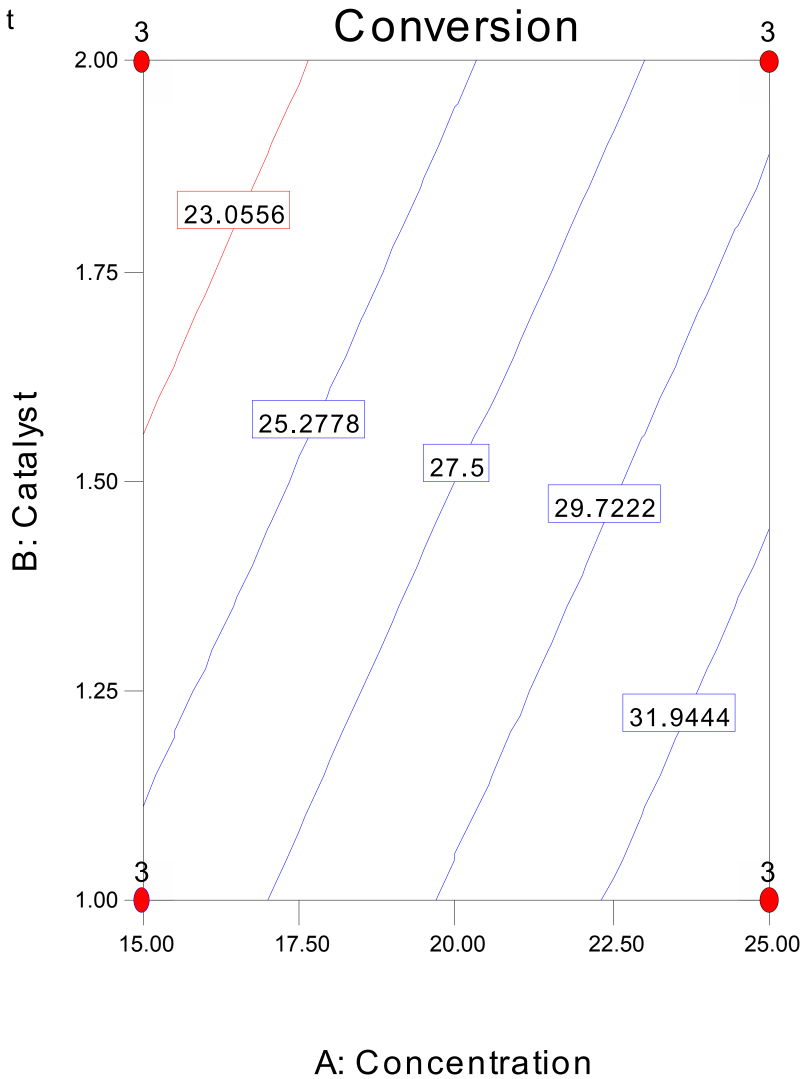
	Coefficient		Standard	95% CI	95% CI	
Factor	Estimate	DF	Error	Low	High	VIF
Intercept	27.5	1	0.60604	26.12904	28.87096	
A-Concent	4.166667	1	0.60604	2.79571	5.537623	1
B-Catalyst	-2.5	1	0.60604	-3.87096	-1.12904	1
Final Equation in Terms of Coded Factors:						
	Conversion	=				
	27.5					
	4.166667	* A				
	-2.5	* B				
Final Equation in Terms of Actual Factors:						
	Conversion	=				
	18.33333					
	0.833333	* Concentration				
	-5	* Catalyst				

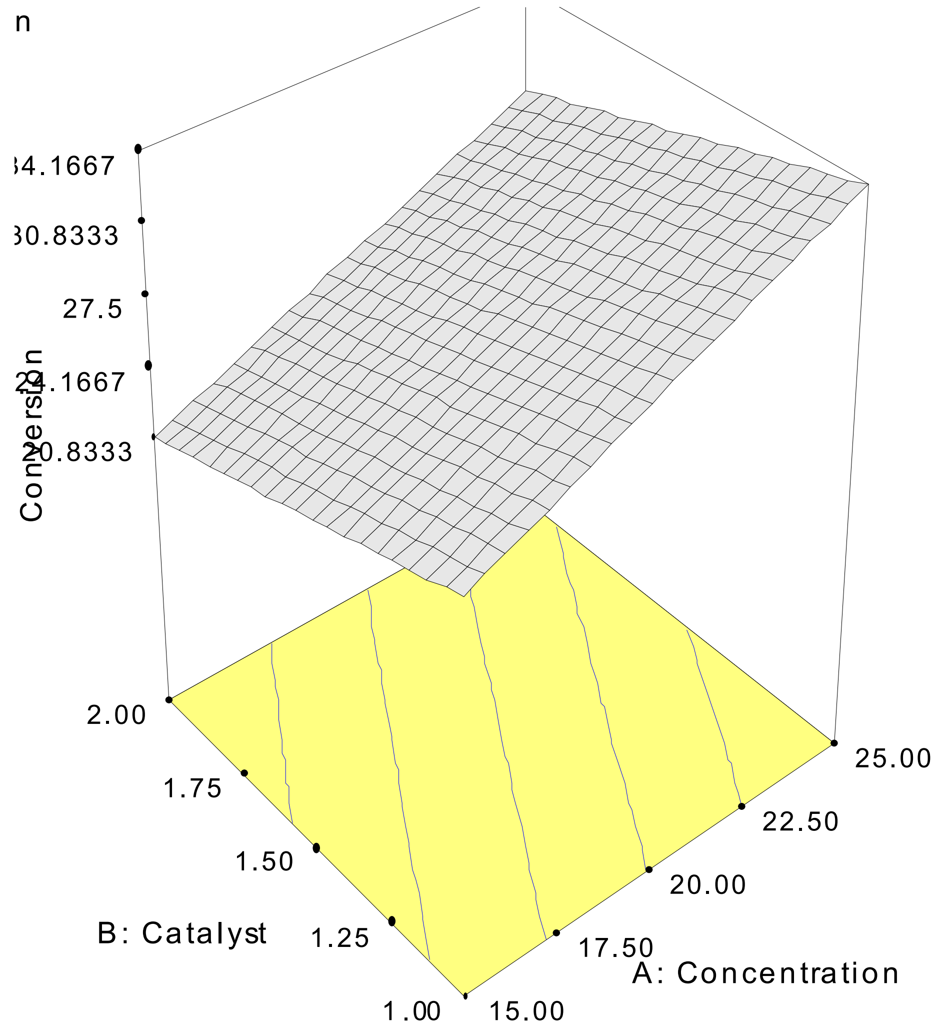
Residuals and Diagnostic Checking



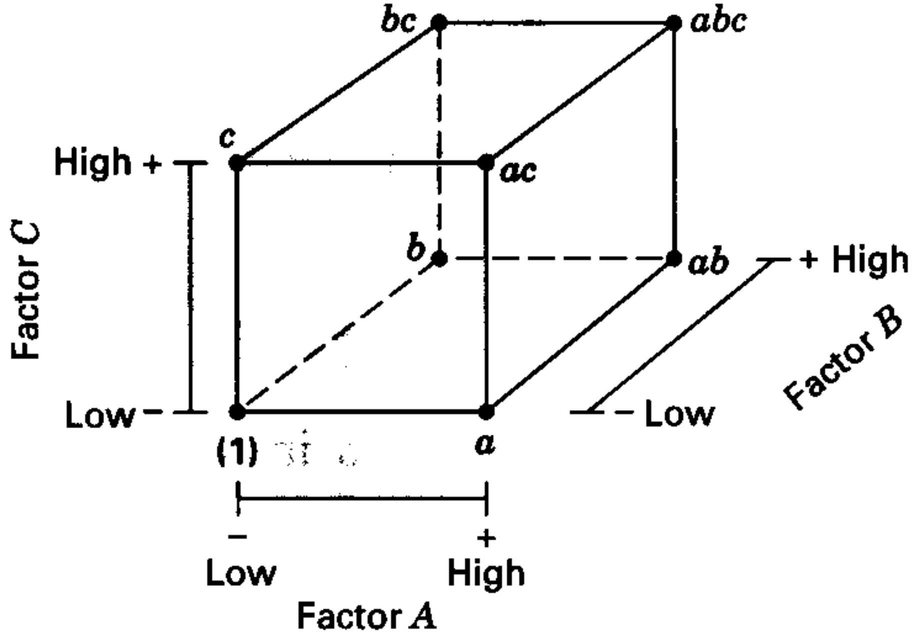


The Response Surface





The 2^3 Factorial Design



(a) Geometric view

Run	Factor A	B	C
1	−	−	−
2	+	−	−
3	−	+	−
4	+	+	−
5	−	−	+
6	+	−	+
7	−	+	+
8	+	+	+

(b) The design matrix

Figure 6-4 The 2^3 factorial design.

Effects in The 2^3 Factorial Design

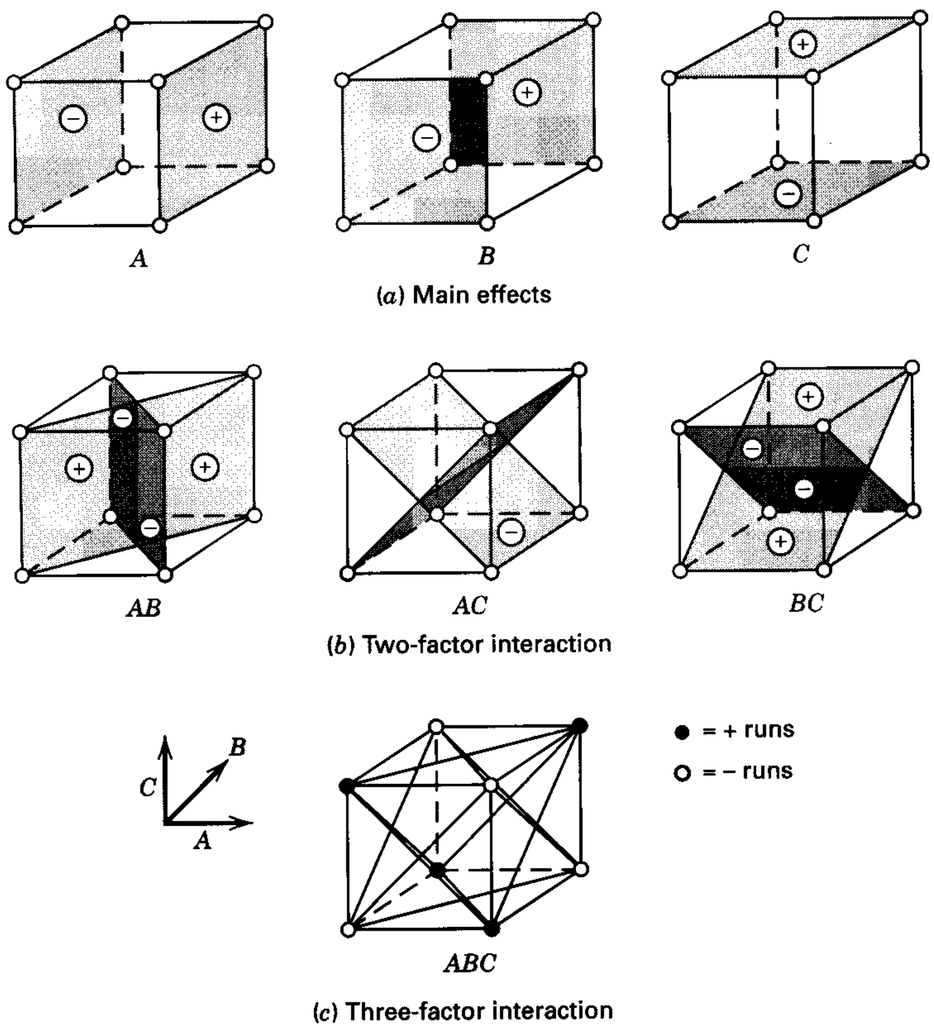


Figure 6-5 Geometric presentation of contrasts corresponding to the main effects and interactions in the 2^3 design.

$$A = \bar{y}_{A^+} - \bar{y}_{A^-}$$

$$B = \bar{y}_{B^+} - \bar{y}_{B^-}$$

$$C = \bar{y}_{C^+} - \bar{y}_{C^-}$$

etc, etc, ...

Analysis done via computer

An Example of a 2^3 Factorial Design

Table 6-4 The Fill Height Experiment, Example 6-1

	Coded Factors			Fill Height Deviation		Factor Levels		
Run	*A*	*B*	*C*	Replicate 1	Replicate 2		Low (−1)	High (+1)
1	−1	−1	−1	−3	−1	*A* (psi)	10	12
2	1	−1	−1	0	1	*B* (psi)	25	30
3	−1	1	−1	−1	0	*C* (b/min)	200	250
4	1	1	−1	2	3			
5	−1	−1	1	−1	0			
6	1	−1	1	2	1			
7	−1	1	1	1	1			
8	1	1	1	6	5			

A = carbonation, B = pressure, C = speed, y = fill deviation

Table of – and + Signs for the 2^3 Factorial Design (pg. 231)

Factorial Effect								
Treatment Combination	***I***	*A*	*B*	*AB*	*C*	*AC*	*BC*	*ABC*
(1) = -4	+	-	-	+	-	+	+	-
a = 1	+	+	-	-	-	-	+	+
b = -1	+	-	+	-	-	+	-	+
ab = 5	+	+	+	+	-	-	-	-
c = -1	+	-	-	+	+	-	-	+
ac = 3	+	+	-	-	+	+	-	-
bc = 2	+	-	+	-	+	-	+	-
abc = 11	+	+	+	+	+	+	+	+
Contrast		24	18	6	14	2	4	4
Effect		3.00	2.25	0.75	1.75	0.25	0.50	0.50

Properties of the Table

- Except for column I, every column has an equal number of + and – signs
- The sum of the product of signs in any two columns is zero
- Multiplying any column by I leaves that column unchanged (identity element)
- The product of any two columns yields a column in the table:

$$A \times B = AB$$

$$AB \times BC = AB^2C = AC$$

- **Orthogonal** design
- Orthogonality is an important property shared by all factorial designs

Estimation of Factor Effects

	Term	Effect	SumSqr	% Contribution
Model	Intercept			
Error	A	3	36	46.1538
Error	B	2.25	20.25	25.9615
Error	C	1.75	12.25	15.7051
Error	AB	0.75	2.25	2.88462
Error	AC	0.25	0.25	0.320513
Error	BC	0.5	1	1.28205
Error	ABC	0.5	1	1.28205
Error	LOF	0		
Error	P Error		5	6.41026

Lenth's ME	1.25382
Lenth's SME	1.88156

ANOVA Summary – Full Model

Response: Fill-deviation

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**
Model	73.00	7	10.43	16.69	0.0003
A	*36.00*	*1*	*36.00*	*57.60*	*< 0.0001*
B	*20.25*	*1*	*20.25*	*32.40*	*0.0005*
C	*12.25*	*1*	*12.25*	*19.60*	*0.0022*
AB	*2.25*	*1*	*2.25*	*3.60*	*0.0943*
AC	*0.25*	*1*	*0.25*	*0.40*	*0.5447*
BC	*1.00*	*1*	*1.00*	*1.60*	*0.2415*
ABC	*1.00*	*1*	*1.00*	*1.60*	*0.2415*
Pure Error	5.00	8	0.63		
Cor Total	78.00	15			

Std. Dev.	0.79	R-Squared	0.9359
Mean	1.00	Adj R-Squared	0.8798
C.V.	79.06	Pred R-Squared	0.7436
PRESS	20.00	Adeq Precision	13.416

Model Coefficients – Full Model

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	1.00	1	0.20	0.54	1.46	
A-Carbonation	1.50	1	0.20	1.04	1.96	1.00
B-Pressure	1.13	1	0.20	0.67	1.58	1.00
C-Speed	0.88	1	0.20	0.42	1.33	1.00
AB	0.38	1	0.20	-0.081	0.83	1.00
AC	0.13	1	0.20	-0.33	0.58	1.00
BC	0.25	1	0.20	-0.21	0.71	1.00
ABC	0.25	1	0.20	-0.21	0.71	1.00

Refine Model – Remove Nonsignificant Factors

Response: Fill-deviation

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	70.75	4	17.69	26.84	< 0.0001
A	*36.00*	*1*	*36.00*	*54.62*	*< 0.0001*
B	*20.25*	*1*	*20.25*	*30.72*	*0.0002*
C	*12.25*	*1*	*12.25*	*18.59*	*0.0012*
AB	*2.25*	*1*	*2.25*	*3.41*	*0.0917*
Residual	7.25	11	0.66		
LOF	*2.25*	*3*	*0.75*	*1.20*	*0.3700*
Pure E	*5.00*	*8*	*0.63*		
C Total	78.00	15			

Std. Dev.	0.81	R-Squared	0.9071
Mean	1.00	Adj R-Squared	0.8733
C.V.	81.18	Pred R-Squared	0.8033
PRESS	15.34	Adeq Precision	15.424

Model Coefficients – Reduced Model

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High
Intercept	1.00	1	0.20	0.55	1.45
A-Carbonation	1.50	1	0.20	1.05	1.95
B-Pressure	1.13	1	0.20	0.68	1.57
C-Speed	0.88	1	0.20	0.43	1.32
AB	0.38	1	0.20	-0.072	0.82

Model Summary Statistics (pg. 239)

- R^2 and adjusted R^2

$$R^2 = \frac{SS_{Model}}{SS_T} = \frac{73.00}{78.00} = 0.9359$$

$$R^2_{Adj} = 1 - \frac{SS_E / df_E}{SS_T / df_T} = 1 - \frac{5.00/8}{78.00/15} = 0.8798$$

- R^2 for prediction (based on PRESS)

$$R^2_{\text{Pred}} = 1 - \frac{PRESS}{SS_T} = 1 - \frac{20.00}{78.00} = 0.7436$$

Model Summary Statistics (pg. 239)

- **Standard error** of model coefficients

$$se(\hat{\beta}) = \sqrt{V(\hat{\beta})} = \sqrt{\frac{\sigma^2}{n2^k}} = \sqrt{\frac{MS_E}{n2^k}} = \sqrt{\frac{0.625}{2(8)}} = 0.20$$

- **Confidence interval** on model coefficients

$$\hat{\beta} - t_{\alpha/2, df_E} se(\hat{\beta}) \le \beta \le \hat{\beta} + t_{\alpha/2, df_E} se(\hat{\beta})$$

The Regression Model

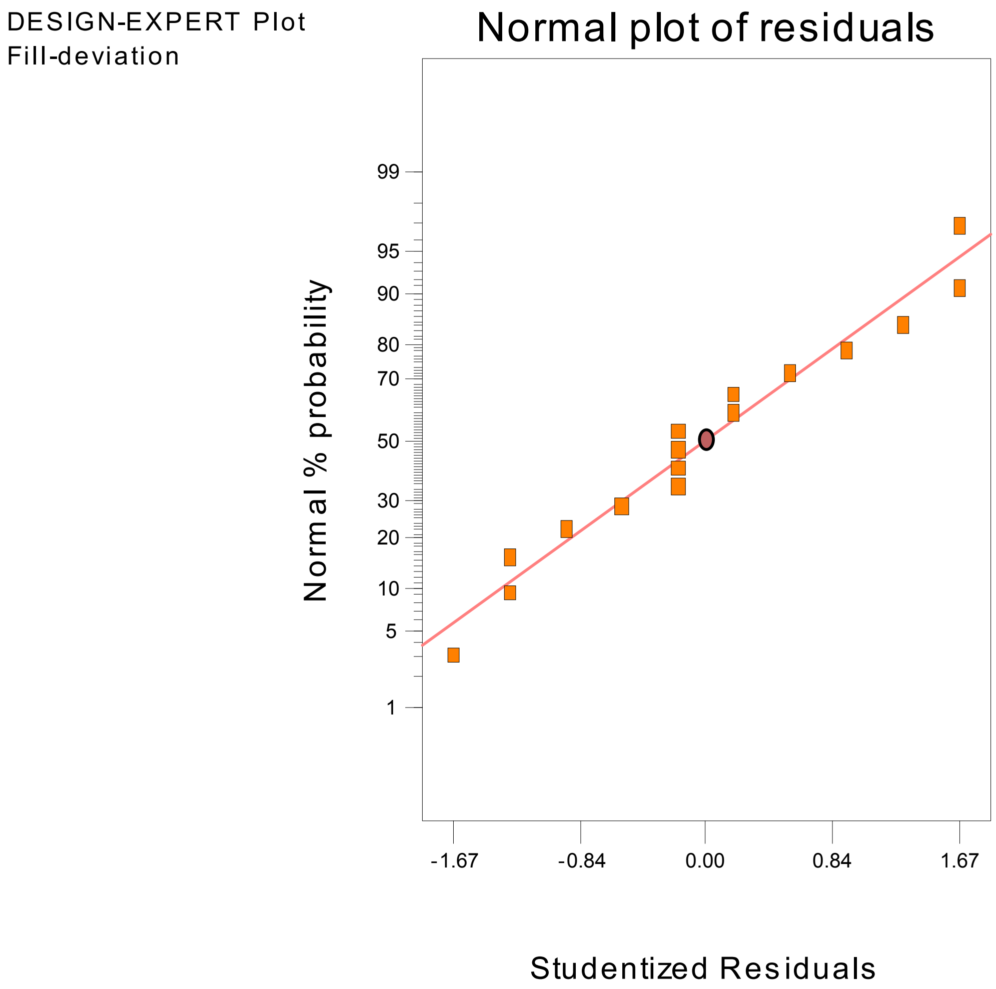
Final Equation in Terms of Coded Factors:

Fill-deviation	=
+1.00	
+1.50	* A
+1.13	* B
+0.88	* C
+0.38	* A * B

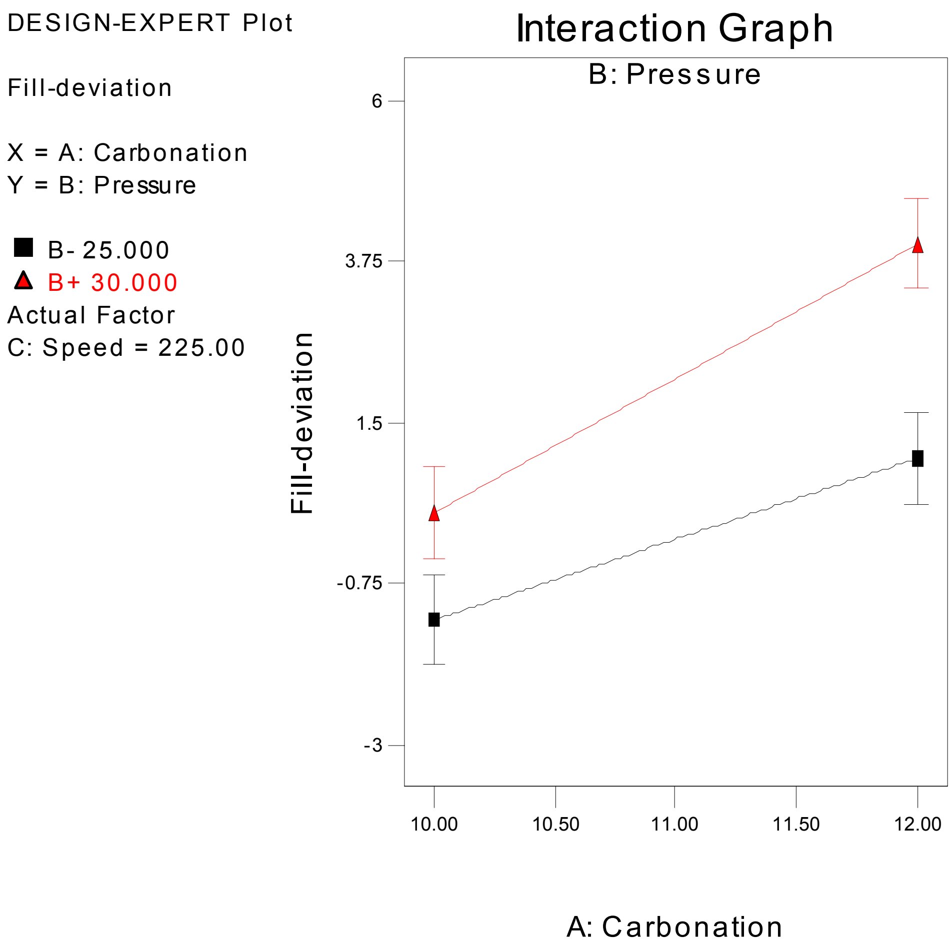
Final Equation in Terms of Actual Factors:

Fill-deviation	=
+9.62500	
-2.62500	* Carbonation
-1.20000	* Pressure
+0.035000	* Speed
+0.15000	* Carbonation * Pressure

Residual Plots are Satisfactory

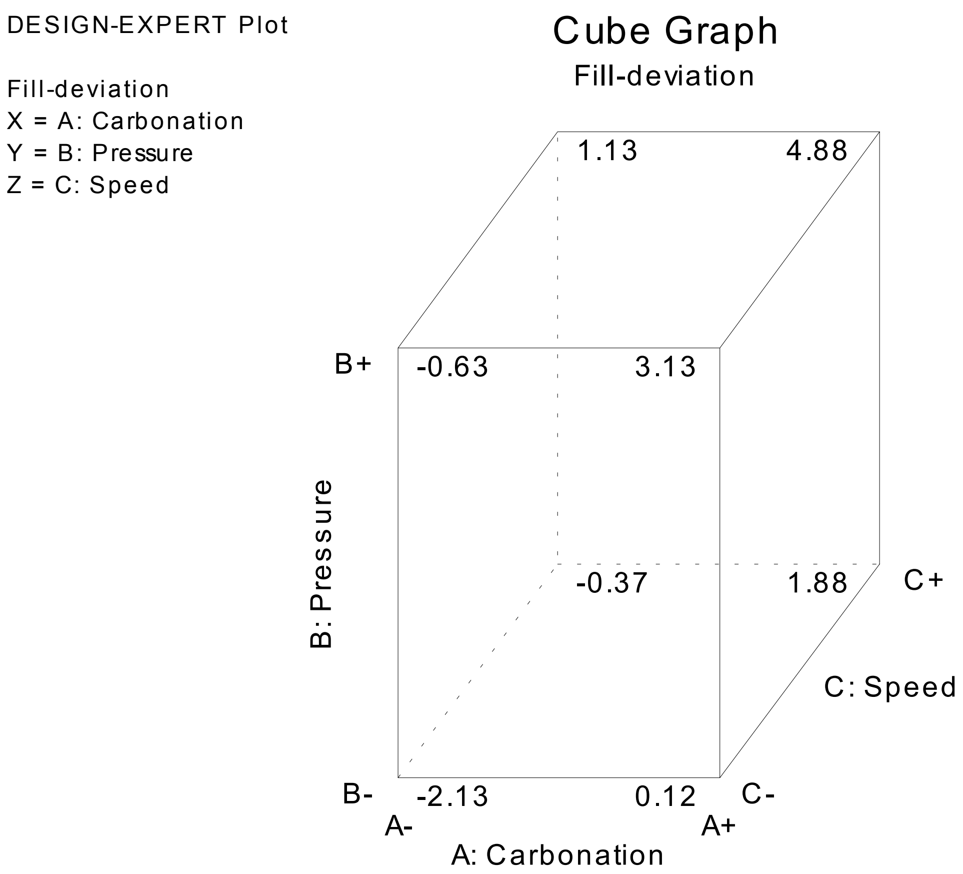


Model Interpretation



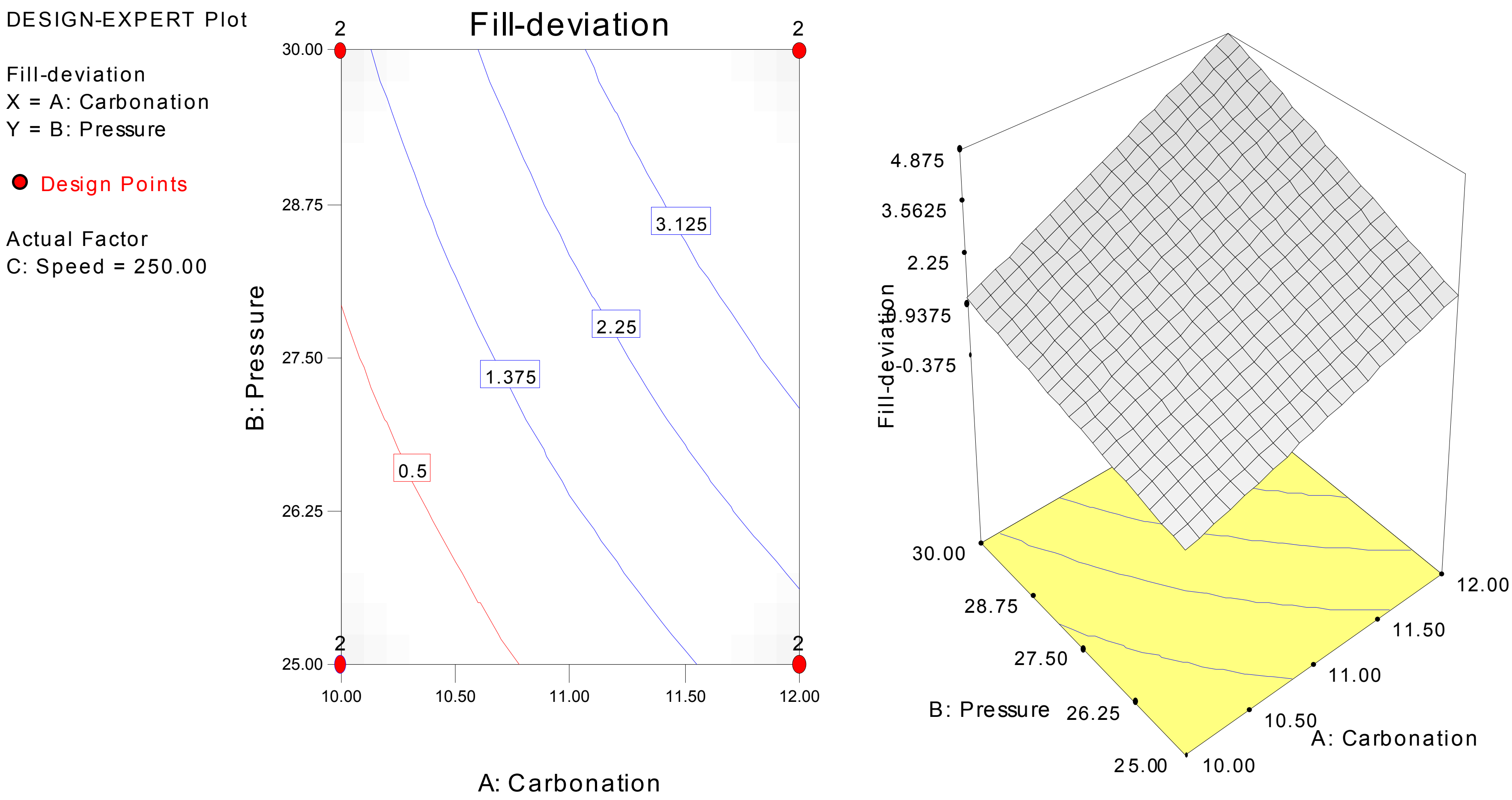
Moderate interaction between carbonation level and pressure

Model Interpretation



Cube plots are often useful visual displays of experimental results

Contour & Response Surface Plots – Speed at the High Level



The General 2^k Factorial Design

- Section 6-4, pg. 242, Table 6-9, pg. 243
- There will be k main effects, and

$\binom{k}{2}$ two-factor interactions

$\binom{k}{3}$ three-factor interactions

$\vdots$

$1\ k-$factor interaction

Unreplicated 2^k Factorial Designs

- These are 2^k factorial designs with **one observation** at each corner of the "cube"
- An unreplicated 2^k factorial design is also sometimes called a "**single replicate**" of the 2^k
- These designs are very widely used
- Risks…if there is only one observation at each corner, is there a chance of unusual response observations spoiling the results?
- Modeling "noise"?

Spacing of Factor Levels in the Unreplicated 2^k Factorial Designs

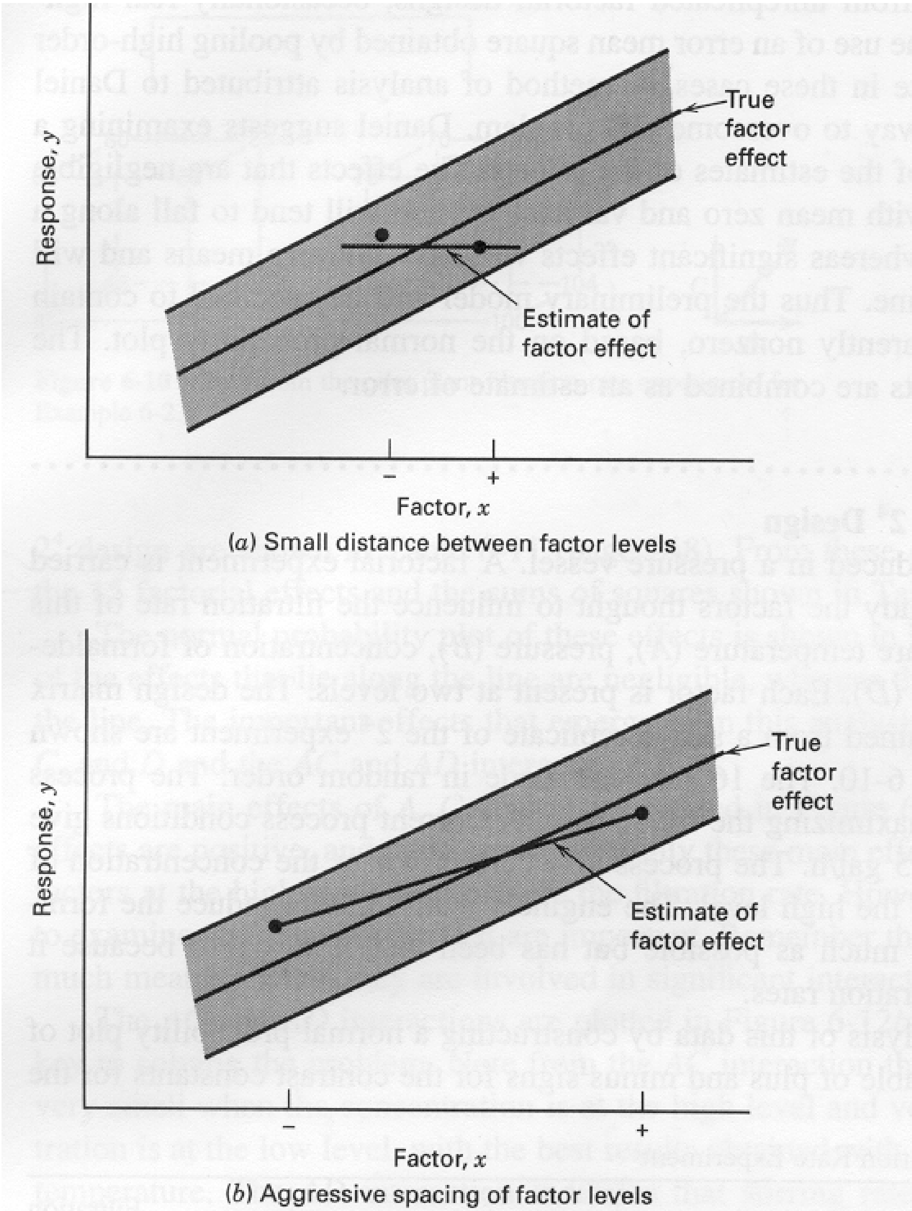


Figure 6-9 The impact of the choice of factor levels in an unreplicated design.

If the factors are spaced too closely, it increases the chances that the noise will overwhelm the signal in the data

More aggressive spacing is usually best

Unreplicated 2^k Factorial Designs

- Lack of replication causes potential **problems** in statistical testing
  - Replication admits an estimate of "pure error" (a better phrase is an **internal estimate** of error)
  - With no replication, fitting the full model results in zero degrees of freedom for error
- Potential **solutions** to this problem
  - Pooling high-order interactions to estimate error
  - **Normal probability plotting** of effects (Daniels, 1959)
  - Other methods…see text, pp. 246

Example of an Unreplicated 2^k Design

- A 2^4 factorial was used to investigate the effects of four factors on the filtration rate of a resin
- The factors are A = temperature, B = pressure, C = mole ratio, D= stirring rate
- Experiment was performed in a pilot plant

The Resin Plant Experiment

Table 6-10 Pilot Plant Filtration Rate Experiment

Run Number	Factor A	B	C	D	Run Label	Filtration Rate (gal/h)
1	−	−	−	−	(1)	45
2	+	−	−	−	*a*	71
3	−	+	−	−	*b*	48
4	+	+	−	−	*ab*	65
5	−	−	+	−	*c*	68
6	+	−	+	−	*ac*	60
7	−	+	+	−	*bc*	80
8	+	+	+	−	*abc*	65
9	−	−	−	+	*d*	43
10	+	−	−	+	*ad*	100
11	−	+	−	+	*bd*	45
12	+	+	−	+	*abd*	104
13	−	−	+	+	*cd*	75
14	+	−	+	+	*acd*	86
15	−	+	+	+	*bcd*	70
16	+	+	+	+	*abcd*	96

The Resin Plant Experiment

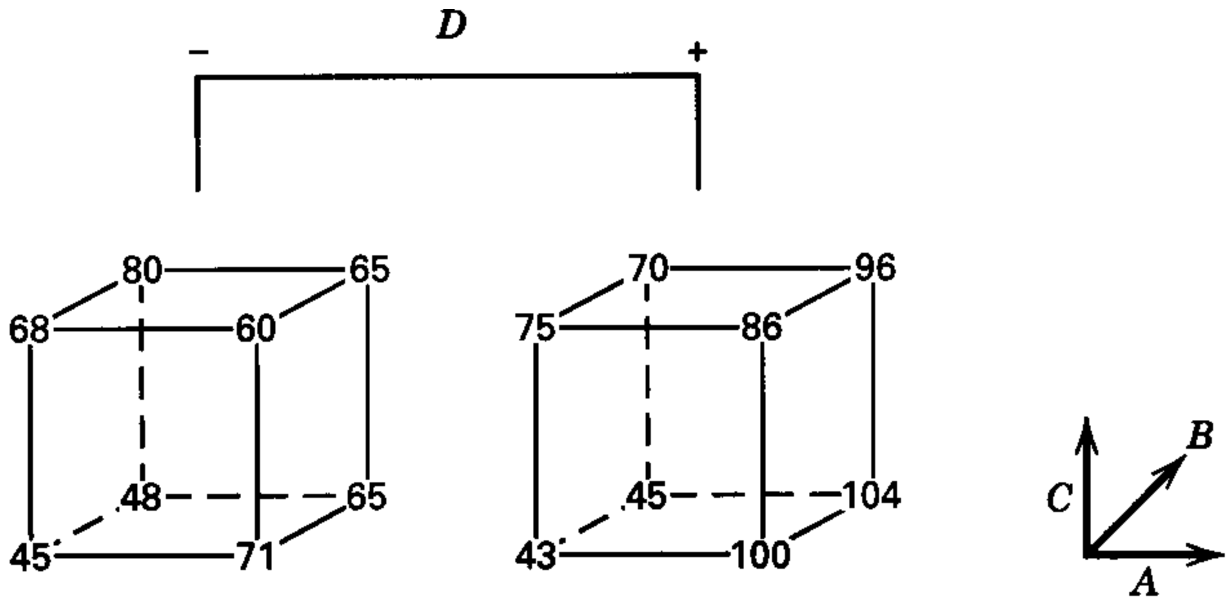
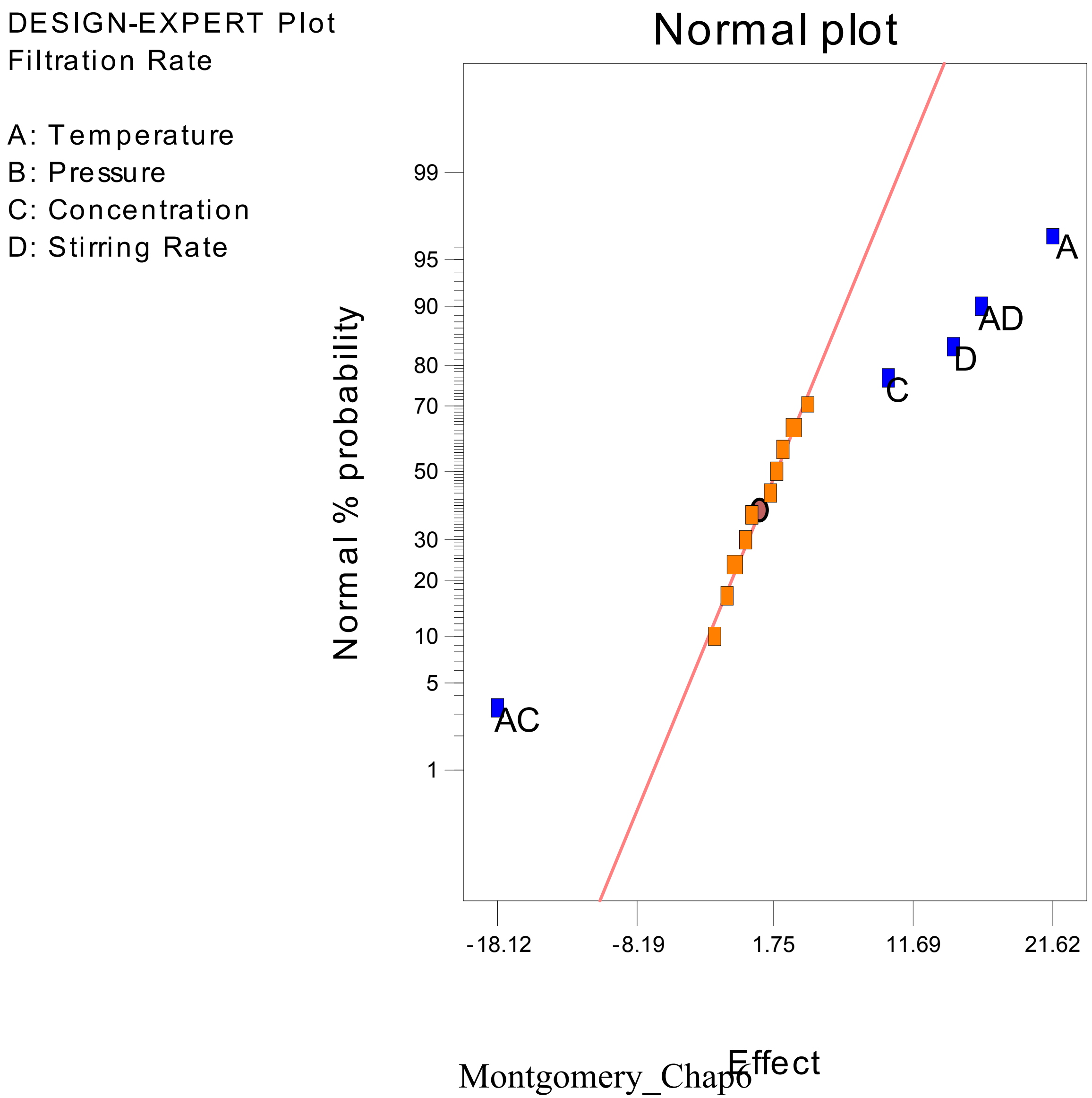


Figure 6-10 Data from the pilot plant filtration rate experiment for Example 6-2.

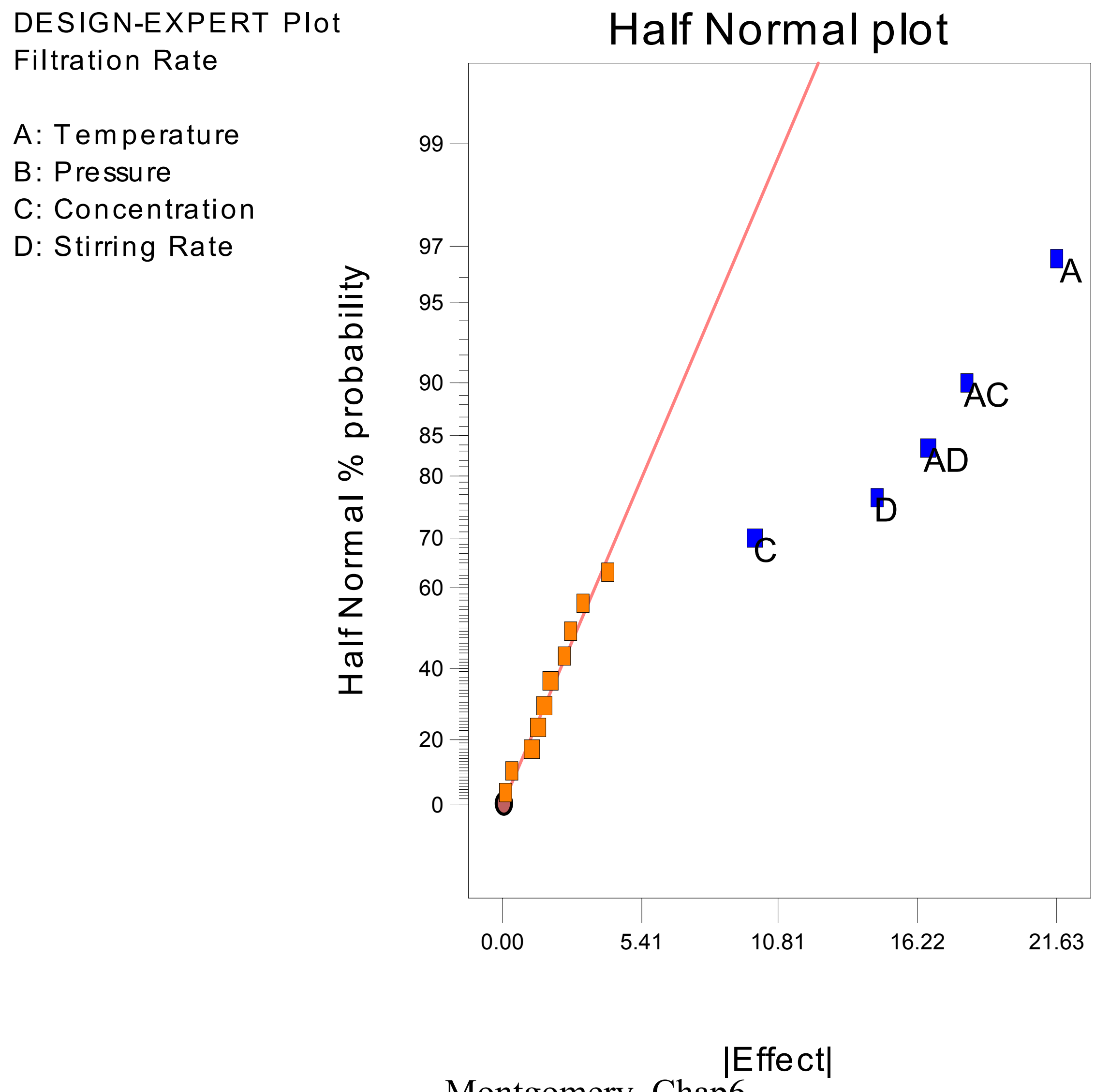
Estimates of the Effects

	Term	Effect	SumSqr	% Contribution
Model	Intercept			
Error	A	21.625	1870.56	32.6397
Error	B	3.125	39.0625	0.681608
Error	C	9.875	390.062	6.80626
Error	D	14.625	855.563	14.9288
Error	AB	0.125	0.0625	0.00109057
Error	AC	-18.125	1314.06	22.9293
Error	AD	16.625	1105.56	19.2911
Error	BC	2.375	22.5625	0.393696
Error	BD	-0.375	0.5625	0.00981515
Error	CD	-1.125	5.0625	0.0883363
Error	ABC	1.875	14.0625	0.245379
Error	ABD	4.125	68.0625	1.18763
Error	ACD	-1.625	10.5625	0.184307
Error	BCD	-2.625	27.5625	0.480942
Error	ABCD	1.375	7.5625	0.131959
	Lenth's ME		6.74778	
	Lenth's SME		13.699	

The Normal Probability Plot of Effects



The Half-Normal Probability Plot



ANOVA Summary for the Model

Response: Filtration Rate
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob >F
Model	5535.81	5	1107.16	56.74	< 0.0001
A	*1870.56*	*1*	*1870.56*	*95.86*	*< 0.0001*
C	*390.06*	*1*	*390.06*	*19.99*	*0.0012*
D	*855.56*	*1*	*855.56*	*43.85*	*< 0.0001*
AC	*1314.06*	*1*	*1314.06*	*67.34*	*< 0.0001*
AD	*1105.56*	*1*	*1105.56*	*56.66*	*< 0.0001*
Residual	195.12	10	19.51		
Cor Total	5730.94	15			

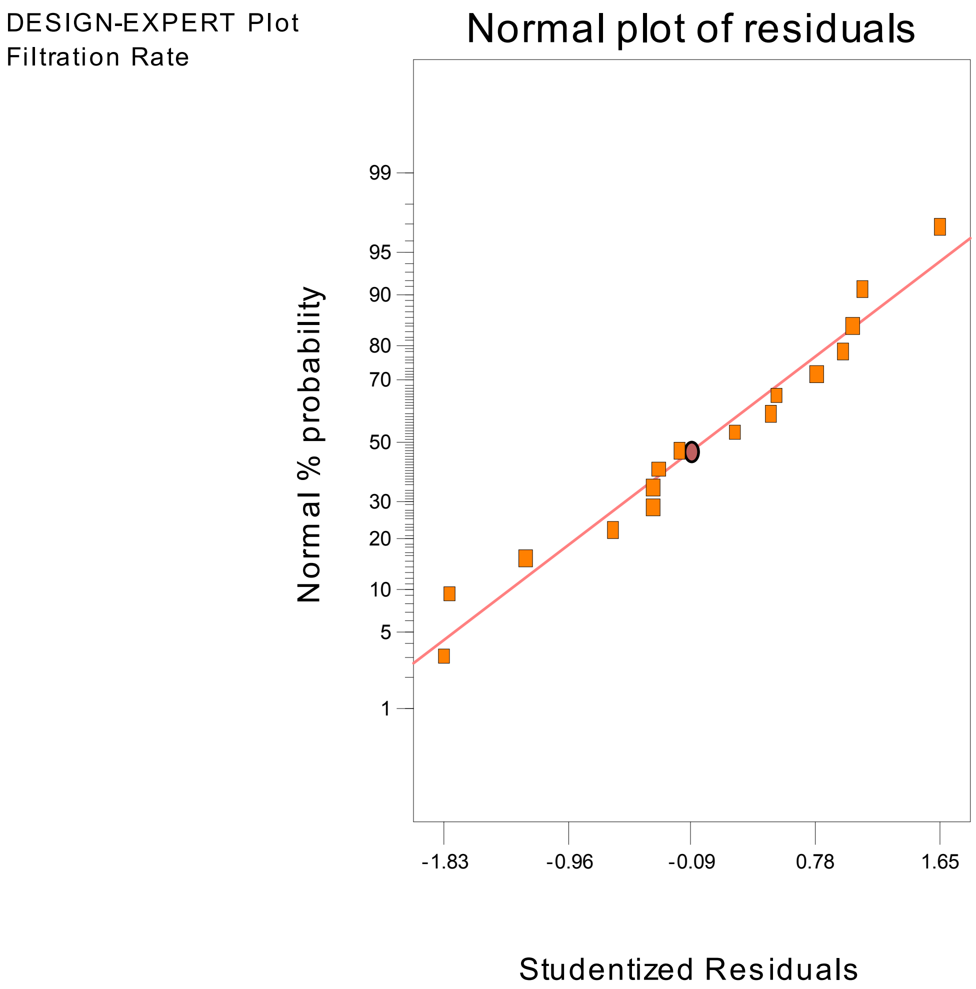
Std. Dev.	4.42	R-Squared	0.9660
Mean	70.06	Adj R-Squared	0.9489
C.V.	6.30	Pred R-Squared	0.9128
PRESS	499.52	Adeq Precision	20.841

The Regression Model

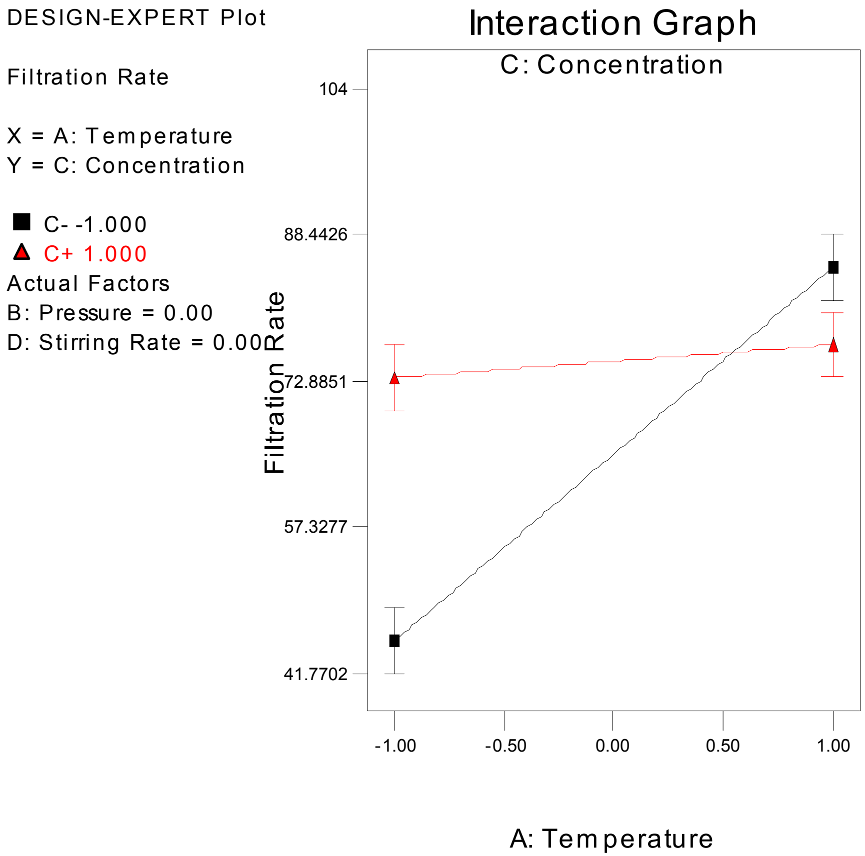
Final Equation in Terms of Coded Factors:

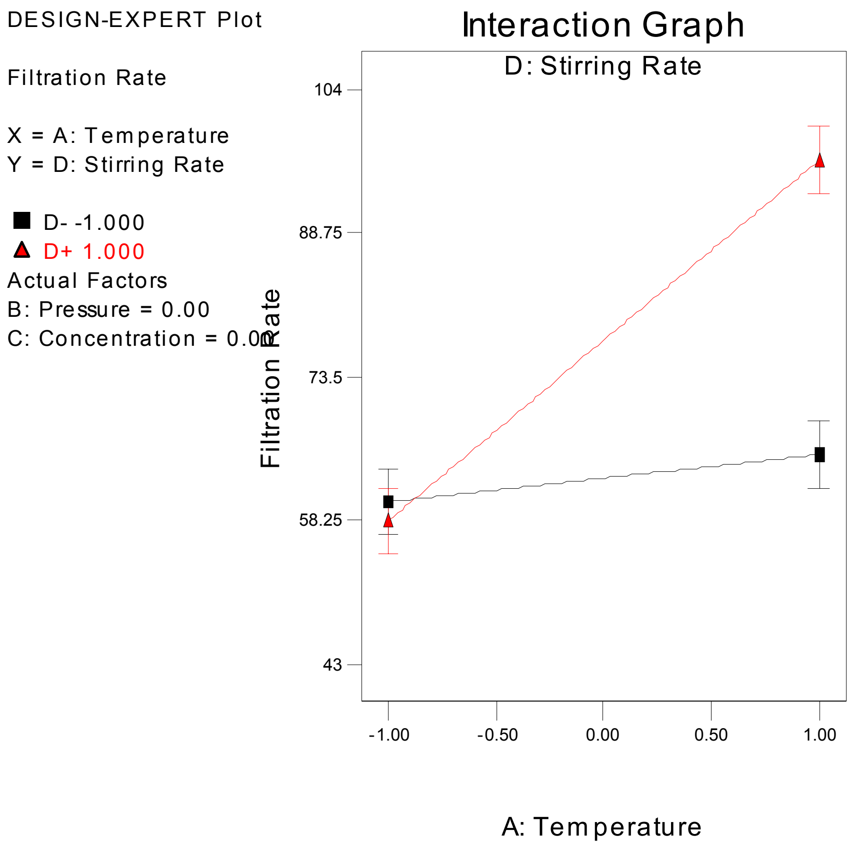
Filtration Rate =
+70.06250
+10.81250 * Temperature
+4.93750 * Concentration
+7.31250 * Stirring Rate
-9.06250 * Temperature * Concentration
+8.31250 * Temperature * Stirring Rate

Model Residuals are Satisfactory

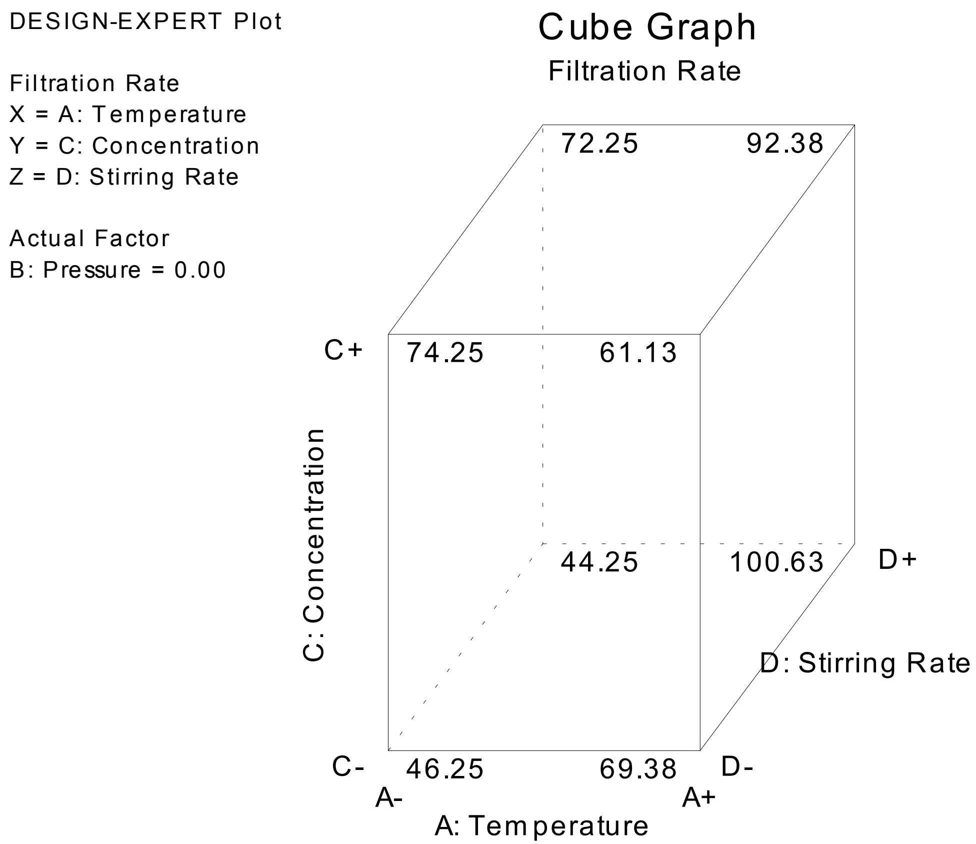


Model Interpretation – Interactions





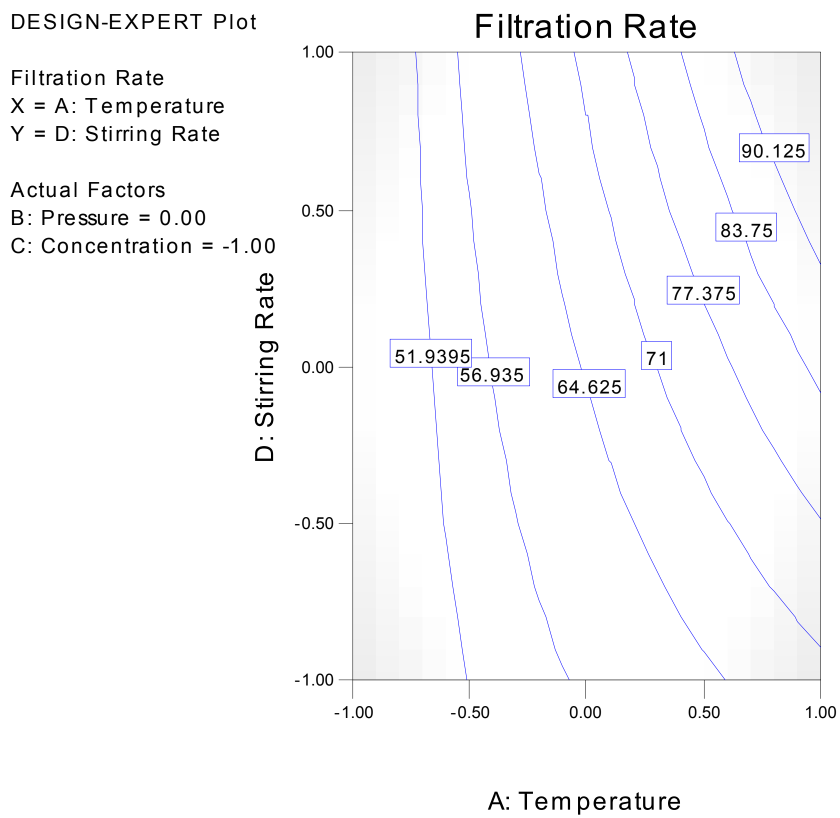
Model Interpretation – Cube Plot

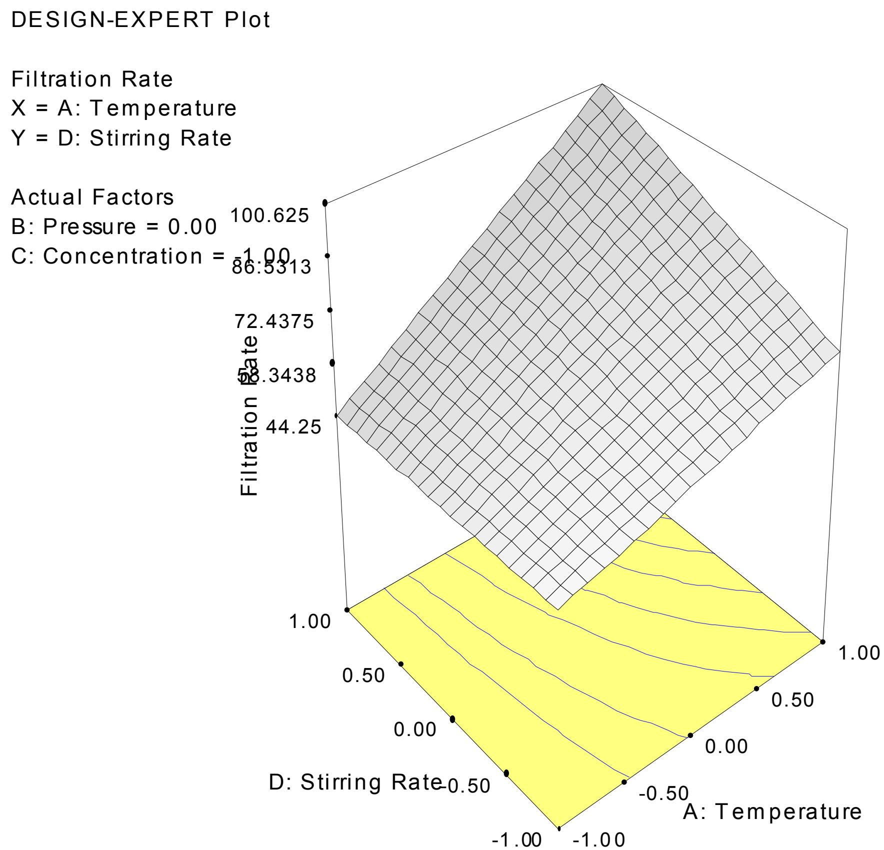


If one factor is dropped, the unreplicated 2^4 design will **project** into two replicates of a 2^3

Design projection is an extremely useful property, carrying over into fractional factorials

Model Interpretation – Response Surface Plots





With concentration at either the low or high level, high temperature and high stirring rate results in high filtration rates

The Drilling Experiment Example 6-3, pg. 257

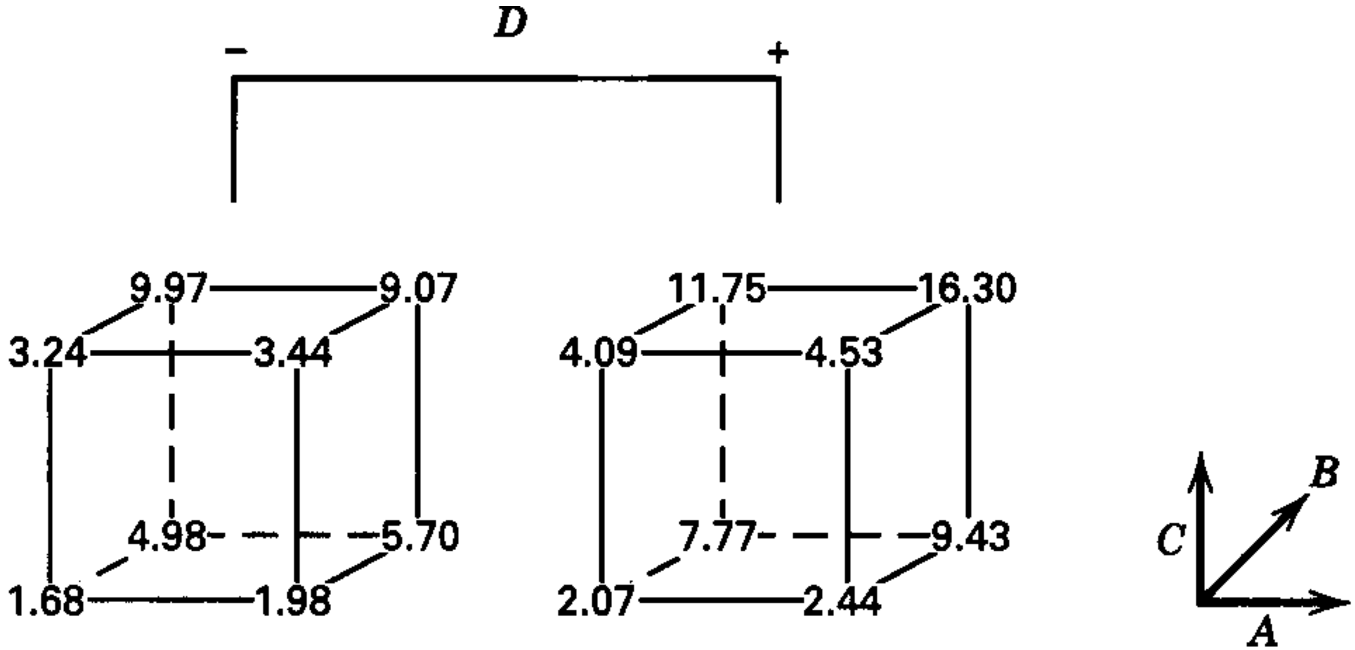


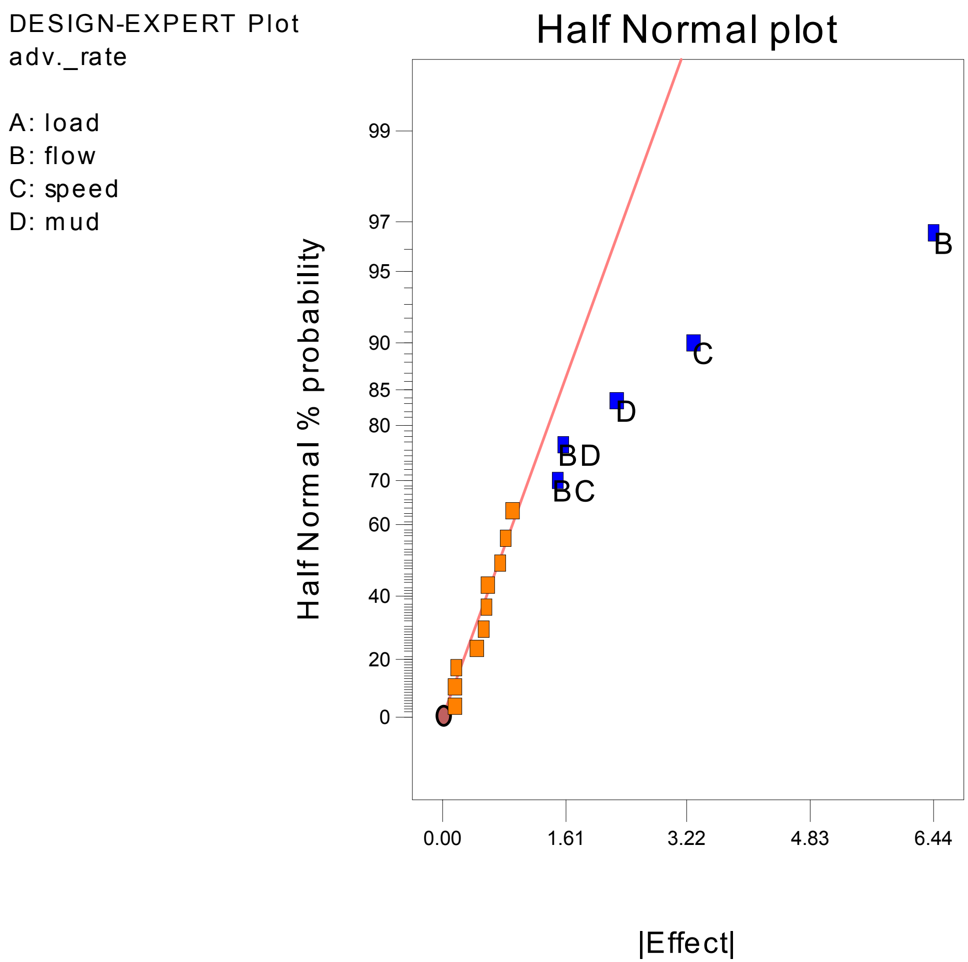
Figure 6-17 Data from the drilling experiment of Example 6-3.

A = drill load, *B* = flow, *C* = speed, *D* = type of mud,
y = advance rate of the drill

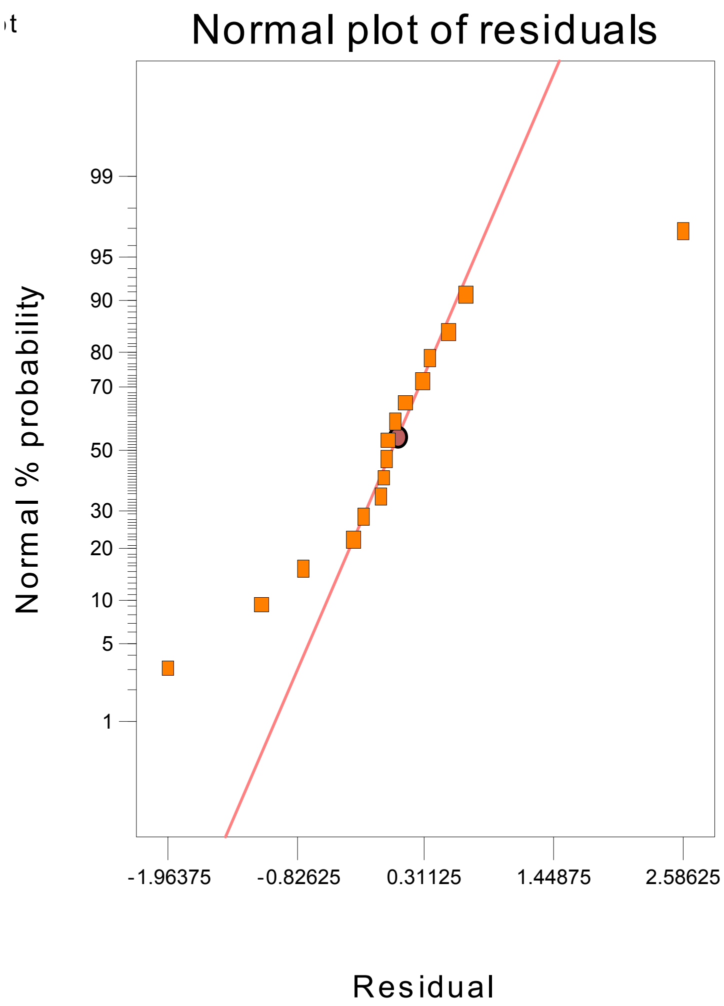
Effect Estimates - The Drilling Experiment

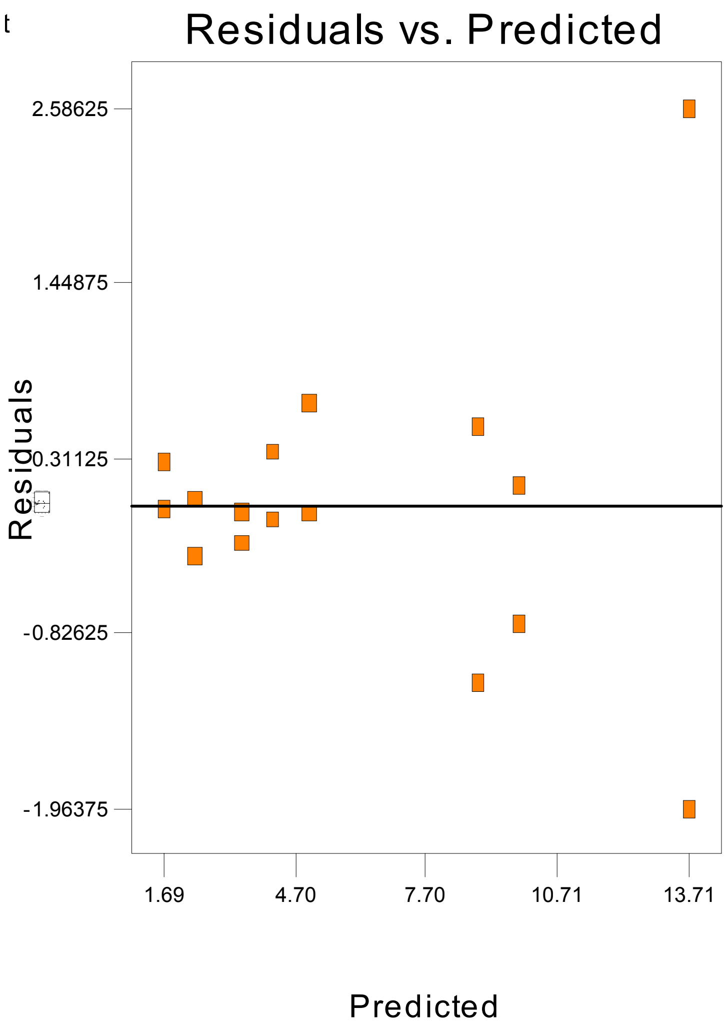
	Term	Effect	SumSqr	% Contribution
Model	Intercept			
Error	A	0.9175	3.36722	1.28072
Error	B	6.4375	165.766	63.0489
Error	C	3.2925	43.3622	16.4928
Error	D	2.29	20.9764	7.97837
Error	AB	0.59	1.3924	0.529599
Error	AC	0.155	0.0961	0.0365516
Error	AD	0.8375	2.80563	1.06712
Error	BC	1.51	9.1204	3.46894
Error	BD	1.5925	10.1442	3.85835
Error	CD	0.4475	0.801025	0.30467
Error	ABC	0.1625	0.105625	0.0401744
Error	ABD	0.76	2.3104	0.87876
Error	ACD	0.585	1.3689	0.520661
Error	BCD	0.175	0.1225	0.0465928
Error	ABCD	0.5425	1.17722	0.447757
	Lenth's ME		2.27496	
	Lenth's SME		4.61851	

Half-Normal Probability Plot of Effects



Residual Plots





Residual Plots

- The residual plots indicate that there are problems with the **equality of variance** assumption
- The usual approach to this problem is to employ a **transformation** on the response
- **Power family** transformations are widely used

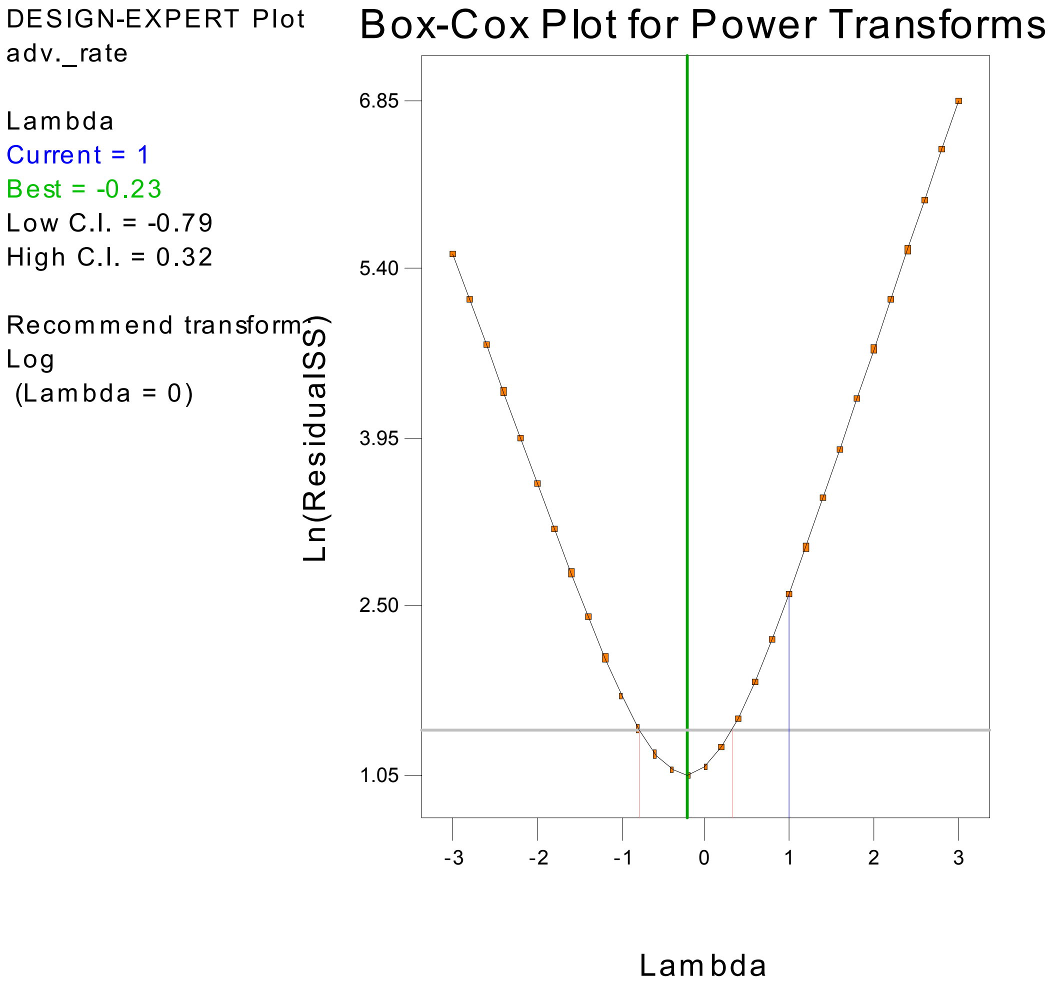
$$y^* = y^\lambda$$

- Transformations are typically performed to
  - Stabilize variance
  - Induce normality
  - Simplify the model

Selecting a Transformation

- **Empirical** selection of lambda
- Prior (theoretical) knowledge or experience can often suggest the form of a transformation
- **Analytical** selection of lambda…the Box-Cox (1964) method (simultaneously estimates the model parameters and the transformation parameter lambda)
- Box-Cox method implemented in Design-Expert

The Box-Cox Method

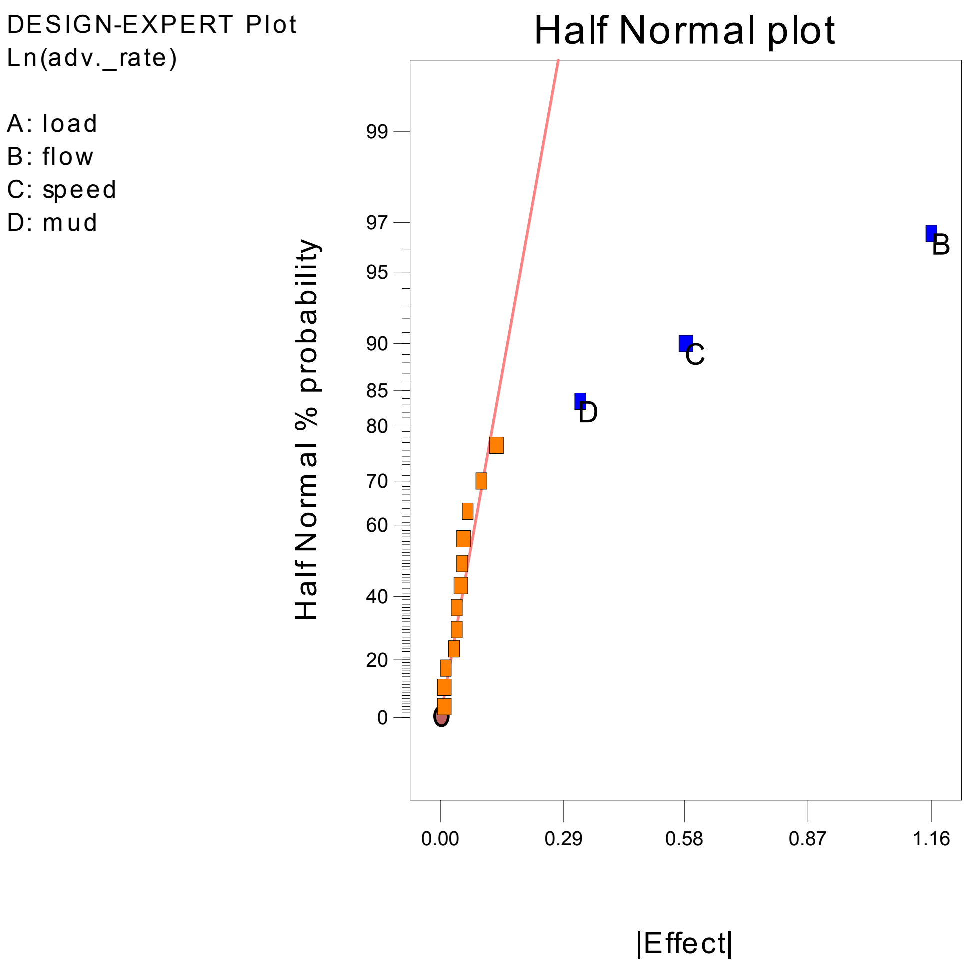


A **log** transformation is recommended

The procedure provides a **confidence interval** on the transformation parameter lambda

If unity is included in the confidence interval, no transformation would be needed

Effect Estimates Following the Log Transformation



Three main effects are large

No indication of large interaction effects

What happened to the interactions?

ANOVA Following the Log Transformation

Response: adv._rate Transform: Natural log
Constant: 0.000

ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**
Model	7.11	3	2.37	164.82	< 0.0001
B	*5.35*	*1*	*5.35*	*371.49*	*< 0.0001*
C	*1.34*	*1*	*1.34*	*93.05*	*< 0.0001*
D	*0.43*	*1*	*0.43*	*29.92*	*0.0001*
Residual	0.17	12	0.014		
Cor Total	7.29	15			

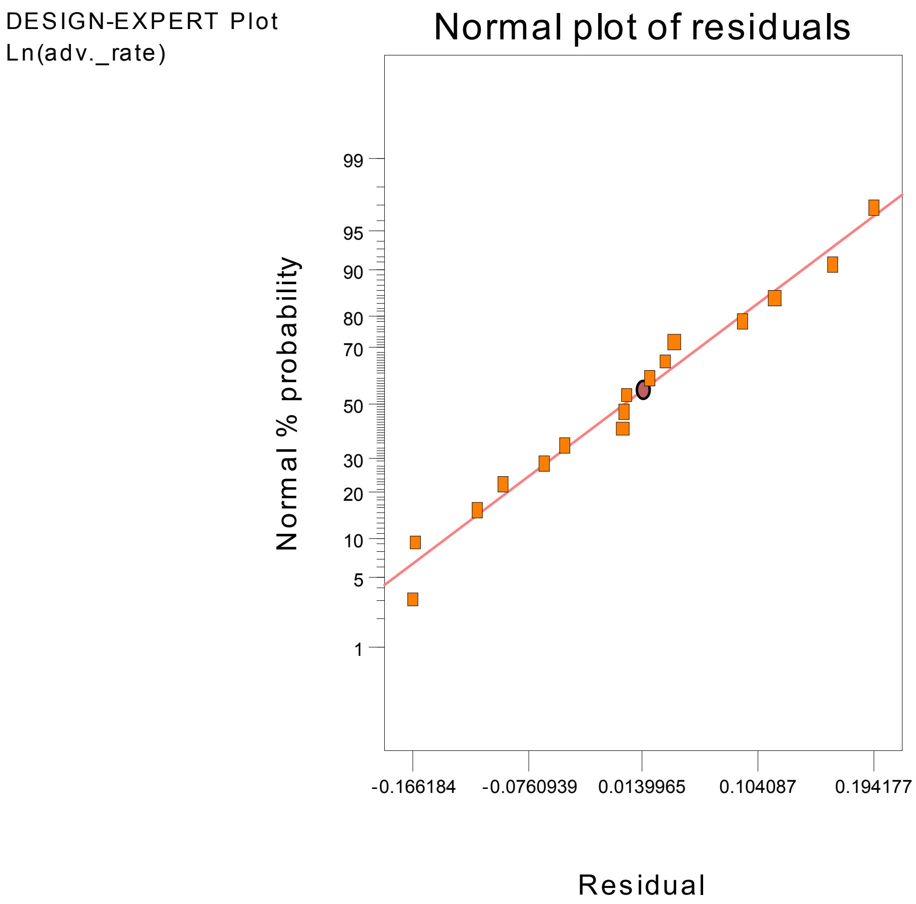
Std. Dev.	0.12	R-Squared	0.9763
Mean	1.60	Adj R-Squared	0.9704
C.V.	7.51	Pred R-Squared	0.9579
PRESS	0.31	Adeq Precision	34.391

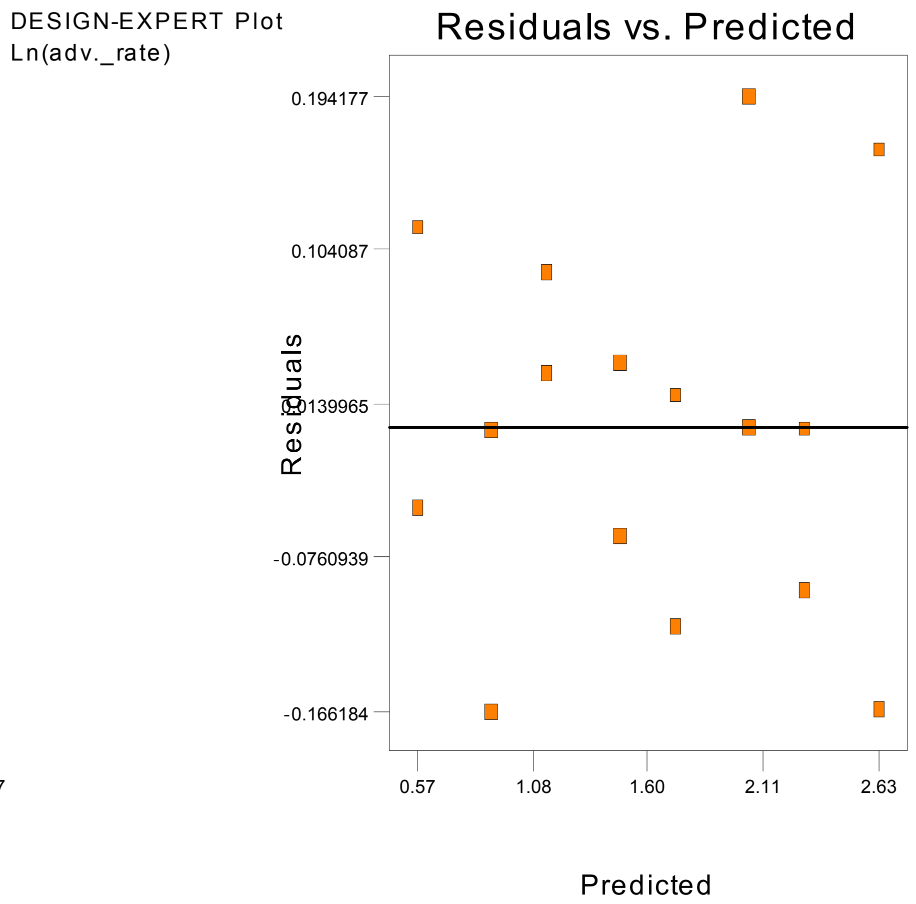
Following the Log Transformation

Final Equation in Terms of Coded Factors:

Ln(adv._rate) =
+1.60
+0.58 * B
+0.29 * C
+0.16 * D

Following the Log Transformation





The Log Advance Rate Model

- Is the log model "better"?
- We would generally prefer a **simpler model** in a transformed scale to a more complicated model in the original metric
- What happened to the interactions?
- Sometimes transformations provide insight into the underlying **mechanism**

Other Examples of Unreplicated 2^k Designs

- The sidewall panel experiment (Example 6-4, pg. 260)
  - Two factors affect the mean number of defects
  - A third factor affects variability
  - Residual plots were useful in identifying the dispersion effect
- The oxidation furnace experiment (Example 6-5, pg. 265)
  - Replicates versus repeat (or duplicate) observations?
  - Modeling within-run variability

Other Analysis Methods for Unreplicated 2^k Designs

- Lenth's method (see text, pg. 254)
  - Analytical method for testing effects, uses an estimate of error formed by pooling small contrasts
  - Some adjustment to the critical values in the original method can be helpful
  - Probably most useful as a supplement to the normal probability plot
- Conditional inference charts (pg. 255 & 256)

Addition of Center Points to a 2^k Designs

- Based on the idea of replicating **some** of the runs in a factorial design
- Runs at the center provide an estimate of error and allow the experimenter to distinguish between two possible models:

First-order model (interaction) $y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \sum_{j>i}^{k} \beta_{ij} x_i x_j + \varepsilon$

Second-order model $y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \sum_{j>i}^{k} \beta_{ij} x_i x_j + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \varepsilon$

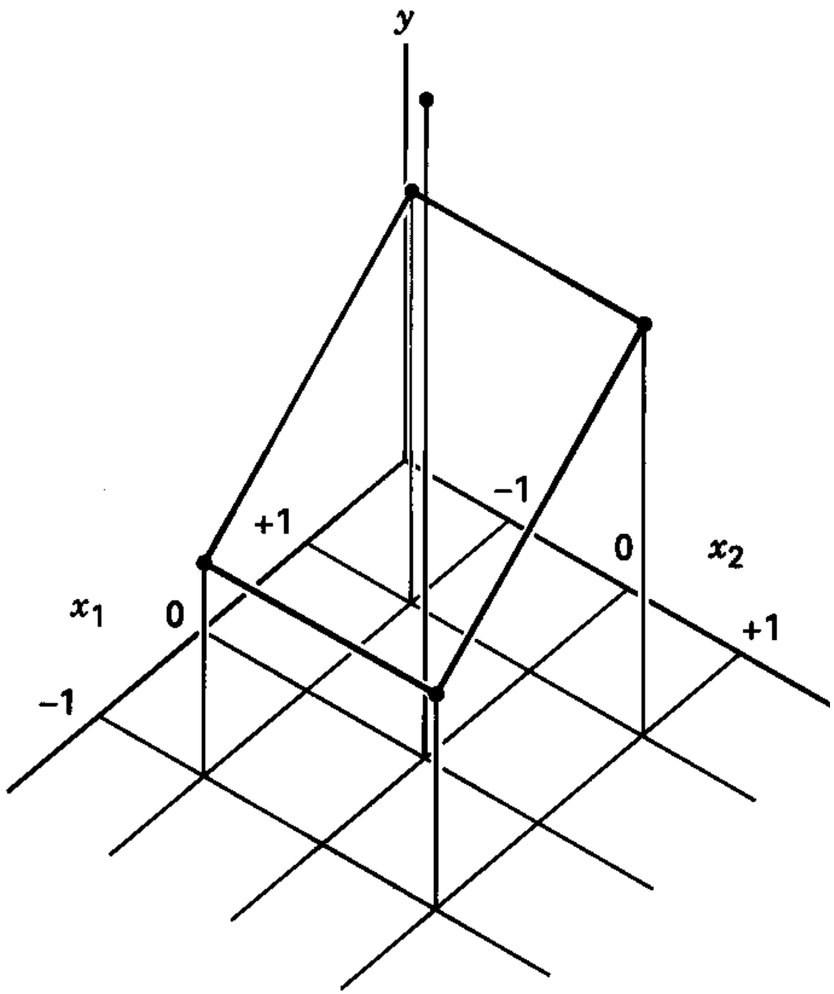


Figure 6-34 A 2^2 design with center points.

$$\bar{y}_F = \bar{y}_C \Rightarrow \text{ no "curvature"}$$

The hypotheses are:

$$H_0 : \sum_{i=1}^{k} \beta_{ii} = 0$$

$$H_1 : \sum_{i=1}^{k} \beta_{ii} \neq 0$$

$$SS_{\text{Pure Quad}} = \frac{n_F n_C (\bar{y}_F - \bar{y}_C)^2}{n_F + n_C}$$

This sum of squares has a single degree of freedom

Example 6-6, Pg. 273

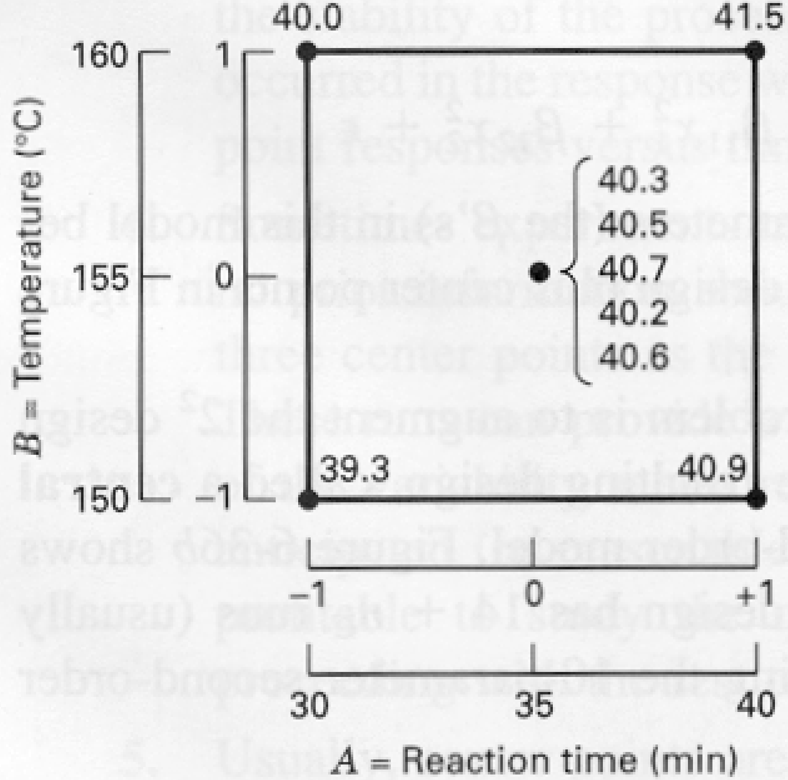


Figure 6-35 The 2^2 design with five center points for Example 6-6.

$$n_C = 5$$

Usually between 3 and 6 center points will work well

Design-Expert provides the analysis, including the *F*-test for pure quadratic curvature

ANOVA for Example 6-6

Response: yield

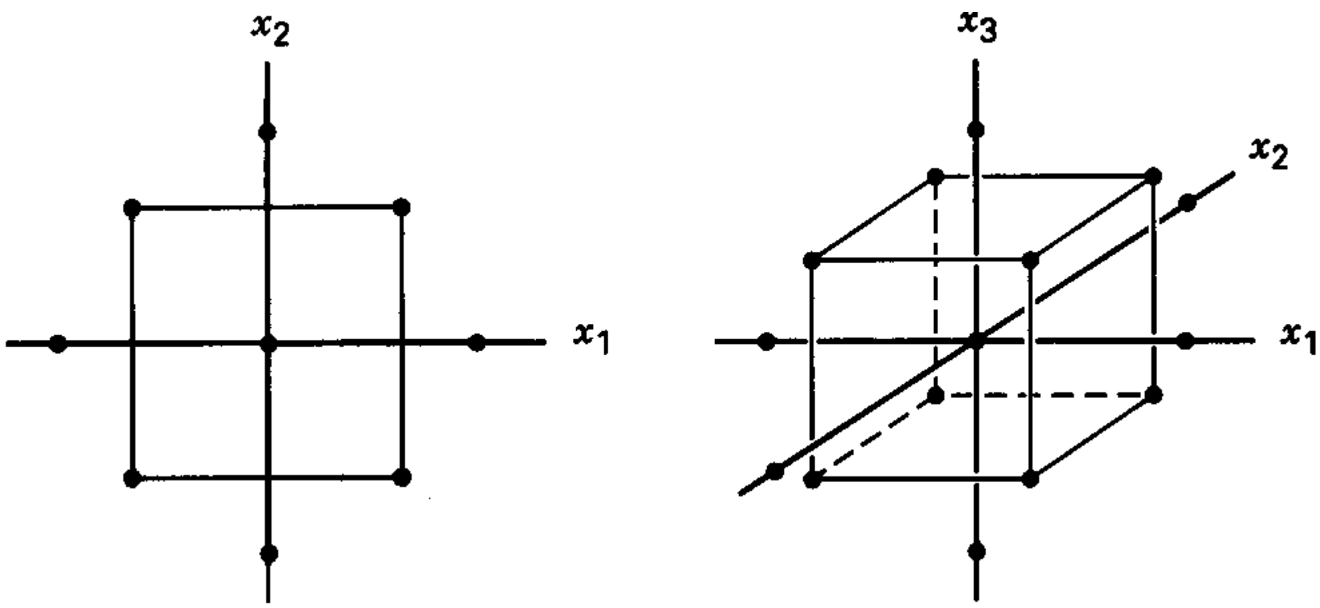
ANOVA for Selected Factorial Model

Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prob > F
Model	2.83	3	0.94	21.92	0.0060
A	*2.40*	*1*	*2.40*	*55.87*	*0.0017*
B	*0.42*	*1*	*0.42*	*9.83*	*0.0350*
AB	*2.500E-003*	*1*	*2.500E-003*	*0.058*	*0.8213*
Curvature	2.722E-003	1	2.722E-003	0.063	0.8137
Pure Error	0.17	4	0.043		
Cor Total	3.00	8			

Std. Dev.	0.21	R-Squared	0.9427
Mean	40.44	Adj R-Squared	0.8996
C.V.	0.51	Pred R-Squared	N/A
PRESS	N/A	Adeq Precision	14.234

If curvature is significant, augment the design with axial runs to create a central composite design. The CCD is a very effective design for fitting a second-order response surface model



(*a*) Two factors (*b*) Three factors

Figure 6-36 Central composite designs.

Practical Use of Center Points (pg. 275)

- Use **current operating conditions** as the center point
- Check for **"abnormal" conditions** during the time the experiment was conducted
- Check for **time trends**
- Use center points as the first few runs when there is little or no information available about the magnitude of **error**
- Center points and **qualitative** factors?

Center Points and Qualitative Factors

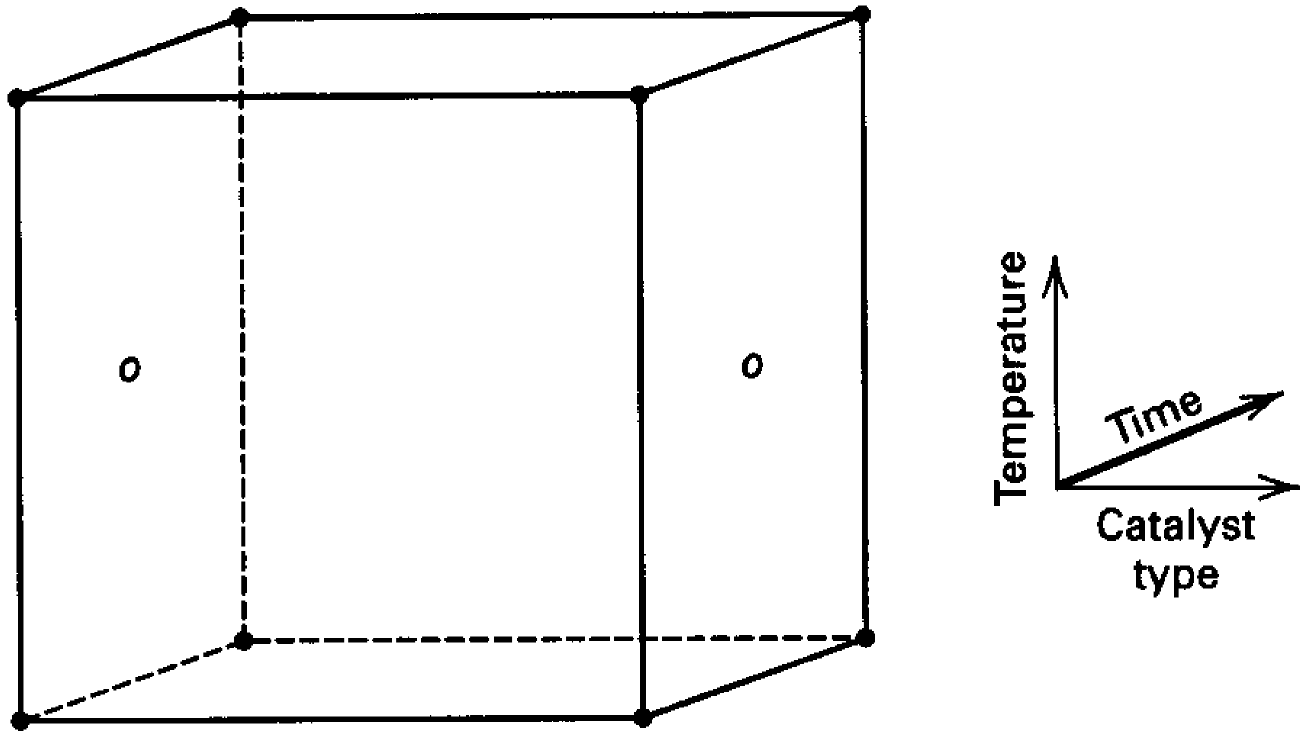


Figure 6-37 A 2^3 factorial design with one qualitative factor and center points.